

IUSEP: Isoperimetric inequality

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Elementary Calculus

- A farmer wishes to enclose a rectangular region R with a fence.
- The fence has total length $L = \text{const} > 0$.
- Show that the area A that can be enclosed obeys

$$A \leq \frac{L^2}{16}$$

and that

$$A = \frac{L^2}{16}$$

if and only if R is a square.

This simple problem has a serious flaw: why should the region R be a rectangle?

- A circle of perimeter L encloses area $A = \pi \cdot \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi} > \frac{L^2}{16}$.

The isoperimetric problem

① In the plane:

- Let $I \subset \mathbb{R}$ be a connected interval of the real line.
- Let $\gamma : I \rightarrow \mathbb{R}^2$ be a simple closed curve of arclength L , enclosing a region R of area A .
- Show that

$$L^2 - 4\pi A \geq 0$$

and that $L^2 = 4\pi A$ if and only if γ is a circle.

② In the 2-sphere of radius a :

- Show that

$$L^2 - 4\pi A + \frac{A^2}{a^2} \geq 0$$

and that $L^2 = 4\pi A$ if and only if γ is a *great circle*.

We will prove the first of these two statements, but first we have to define everything. The key tools are *differential geometry* and *geometric analysis*.

Parametrized curves

Definition

A *parametrized curve* is a map $\gamma : I \rightarrow \mathbb{R}^n$, where I is a connected interval of $\mathbb{R}$.

- Example: It's very easy to parametrize a graph $y = f(x)$.
 - Just choose x to be the parameter; i.e., write

$$\begin{aligned}x(t) &= t, \\ y(t) &= f(t).\end{aligned}$$

- Don't forget to choose domain; for example, perhaps

$$t \in I$$

for some interval $I \subset \mathbb{R}$.

More examples

- The parametrized curve $\begin{cases} x(t) = t, \\ y(t) = \sqrt{a^2 - t^2}, \\ t \in [-a, a], \end{cases}$ is a semi-circle.
- The parametrized curve $\begin{cases} x(t) = \cos t, \\ y(t) = \sin t, \\ t \in [0, 2\pi), \end{cases}$ is a circle, traversed once counter-clockwise.
- The parametrized curve $\begin{cases} x(t) = \cos t, \\ y(t) = \sin t, \\ t \in [0, 4\pi), \end{cases}$ is a circle, traversed twice counter-clockwise.

Notice the parametrization carries extra information not available from the graphical description of a curve.

Example: The astroid

- The parametrized curve $\gamma(t) = (\cos^3 t, \sin^3 t)$, $t \in [0, 2\pi)$, is called an *astroid*.
- Can write it as
$$\begin{cases} x(t) = \cos^3 t \\ y(t) = \sin^3 t \\ t \in [0, 2\pi) \end{cases}$$
- Then $x^{2/3} = \cos^2 t$ and $y^{2/3} = \sin^2 t$,
so $x^{2/3} + y^{2/3} = 1$.
- Graphical form: $y = \pm (1 - x^{2/3})^{3/2}$.
- Level set form:
 - Let $z = f(x, y) = x^{2/3} + y^{2/3}$.
 - Then the astroid is the level set $z = f(x, y) = 1$.
- Graphical and level set forms have less information than parametrized form.

Tangent vectors

- Recall tangent line to graph $y = f(x)$ at (x_0, y_0) is $y - y_0 = f'(x_0)(x - x_0)$.
- Tangent vector: Any (non-zero) vector parallel to tangent line.
- Parametrized form of line: Take $s \in \mathbb{R}$ and

$$\begin{aligned}x(s) &= x_0 + s, \\ y(s) &= y_0 + f'(x_0)s.\end{aligned}$$

- Differentiate with respect to s : $x'(s) = 1$, $y'(s) = f'(x_0)$.
- Tangent vectors to line are the vectors parallel to $(1, f'(x_0))$.

Tangent vectors to parametrized curves

- Parametrized curve $\gamma : I \rightarrow \mathbb{R}^n$ is a *vector-valued* function.
- $\gamma(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t)) = (x_1(t), x_2(t), \dots, x_n(t))$.

Definition

$$\begin{aligned}\gamma'(t) = \dot{\gamma}(t) &= \frac{d\gamma}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \dots, \frac{dx_n}{dt} \right) \\ &= \lim_{\Delta t \rightarrow 0} \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t}\end{aligned}$$

Then $\gamma'(t)$ is a *tangent vector* to curve γ at t provided $\gamma'(t) \neq (0, \dots, 0)$.

(Generally, we will just write 0 even if we mean the 0-vector $(0, 0, \dots, 0)$.)

Example

- $\gamma(t) = t^3 \mathbf{e}_1 + t^2 \mathbf{e}_2$, $t \in \mathbb{R}$, $\{\mathbf{e}_1, \mathbf{e}_2\} =$ orthonormal basis (ONB).
- $\left. \begin{aligned} x(t) &= t^3 \\ y(t) &= t^2 \end{aligned} \right\} \implies y = x^{2/3}.$
- Chain rule:
 - $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \implies 2t = \frac{dy}{dx} \cdot 3t^2 \implies \frac{dy}{dx} = \frac{2t}{3t^2}$ undefined at $t = 0$.

Definition

A parametrized curve $\gamma : I \rightarrow \mathbb{R}^n$ is

- *smooth* at $t_0 \in I$ if all derivatives of all components $x_i(t)$ exist at $t = t_0$, and
- *regular* at $t_0 \in I$ if it is smooth at t_0 and $\frac{d\gamma}{dt}(t_0) \neq (0, \dots, 0)$; otherwise t_0 is a *singular point*.

The above example is smooth but not regular at $t = 0$.

Arclength

- Recall arclength in $\mathbb{R}^2$:

$$s = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- In $\mathbb{R}^n$: $s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2} dt = \int_{t_0}^{t_1} \sqrt{\frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt}} dt = \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt} \right\| dt$

Definition

The arclength function of a curve $\gamma : [t_0, t_1] \rightarrow \mathbb{R}^n$ is

$$s := \int_{t_0}^t \left\| \frac{d\gamma(t')}{dt'} \right\| dt'$$

for $t \in [t_0, t_1]$.

Fundamental Theorem of Calculus $\implies \frac{ds}{dt} = \left\| \frac{d\gamma(t)}{dt} \right\|$.

This is called the *speed* of the curve.

Example: Log spiral

- The logarithmic spiral is the curve

$$\gamma(t) = e^t (\cos t, \sin t).$$

- $\gamma'(t) = e^t (\cos t - \sin t, \sin t + \cos t)$

- $\|\gamma'\| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2}e^t.$

- $s(t) = \int_{t_0}^t \sqrt{2}e^\tau d\tau = \sqrt{2}(e^t - e^{t_0}).$

- $t_0 \rightarrow -\infty \implies \gamma(t_0) \rightarrow (0, 0), s(t) \rightarrow \sqrt{2}e^t.$

- $\gamma : (-\infty, t] \rightarrow \mathbb{R}^2$ has finite arclength, but no initial endpoint.

Unit speed curves

- If $\|\dot{\gamma}(t)\| = 1$, γ is *unit speed* and t is an *arclength parameter* or *unit speed parameter*.
- If $\|\dot{\gamma}(t)\| = k = \text{const} > 0$, γ is *constant speed* and t is an *affine parameter*.
- Fact:
 - Let $\mathbf{v}$ be any unit vector field $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1$.
 - Let $\gamma(t)$ be a unit speed curve.
 - $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d}{dt}(1) = 0$.
 - But then $\frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = 0$.
 - Chain rule: $\dot{\gamma} \cdot \ddot{\gamma} = 0$.
 - Conclude that $\dot{\gamma} \perp \ddot{\gamma}$ along any unit speed curve whenever acceleration $\ddot{\gamma} \neq 0$.
 - For unit speed curves, write $\mathbf{T} := \dot{\gamma} =$ unit tangent vector. Note that $\|\mathbf{T}\| = \sqrt{\mathbf{T} \cdot \mathbf{T}} = 1$.

Reparametrization

- Say $\gamma : (a, b) \rightarrow \mathbb{R}^n$ is a curve, and
- Say $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \rightarrow \mathbb{R}^n$ is a curve.

Definition

If

- there is a smooth map $\phi : (\tilde{a}, \tilde{b}) \rightarrow (a, b)$
- with smooth inverse $\phi^{-1} : (a, b) \rightarrow (\tilde{a}, \tilde{b})$, such that
- $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) = (\gamma \circ \phi)(\tilde{t}) = \gamma(t)$ for all $\tilde{t} \in (\tilde{a}, \tilde{b})$,

then $\tilde{\gamma} = \gamma \circ \phi$ is a *reparametrization* of γ .

Theorem

Theorem

Any reparametrization of a regular curve is also a regular curve.

Proof.

- Let $t = \phi(\tilde{t})$ and $\tilde{\gamma}(\tilde{t}) = \gamma(t)$.
- Then $\tilde{t} = \phi^{-1}(t)$ so $t = \phi(\tilde{t}) = \phi(\phi^{-1}(t))$.
- Chain rule: $\frac{d\phi}{d\tilde{t}} \frac{d(\phi^{-1})}{dt} = 1$, so $\frac{d\phi}{d\tilde{t}} \neq 0$.
- $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d}{d\tilde{t}}(\gamma(t)) = \frac{d\gamma}{dt} \frac{d\phi}{d\tilde{t}}$.
- Now γ is regular so $\frac{d\gamma}{dt} \neq 0$, and $\frac{d\phi}{d\tilde{t}} \neq 0$.
- Thus $\frac{d\tilde{\gamma}}{d\tilde{t}} \neq 0$.



Works because the reparametrization ϕ is smooth with smooth inverse.

The arclength function of a regular curve is smooth

- Say $\gamma : I \rightarrow \mathbb{R}^2 : t \rightarrow (x(t), y(t))$ is a regular curve.
- Then $x(t)$ and $y(t)$ are smooth functions.
- The square root function $f(w) = \sqrt{w}$ is smooth if $w \neq 0$.
- Since γ is regular, $\dot{x}^2 + \dot{y}^2 \neq 0$.
- Thus $\frac{ds}{dt}(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$ is smooth.
- Therefore $s(t) = \int_{t_0}^t \frac{ds}{dt'}(t') dt'$ is smooth.

Regular curves have unit speed parametrizations

Theorem

A parametrized curve has an arclength parametrization iff it is regular.

Proof.

- Curve $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^2$ and reparametrization $t = \phi(\tilde{t})$, such that $\gamma(t) = \tilde{\gamma}(\tilde{t})$.
 - Chain rule: $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt} \frac{dt}{d\tilde{t}} \implies \left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = \left\| \frac{d\gamma}{dt} \right\| \left| \frac{dt}{d\tilde{t}} \right|$.
- $\Rightarrow$ If $\tilde{t}$ is arclength, then $\left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = 1$, so $\frac{d\gamma}{dt}$ is never zero. Then $\gamma(t)$ is regular.
- $\Leftarrow$
- If $\frac{d\gamma}{dt} \neq 0$, then $\frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\| \neq 0$, so s is smooth and strictly increasing.
 - Then $\frac{d\gamma}{dt} = \frac{d\tilde{\gamma}}{ds} \frac{ds}{dt} \implies \left\| \frac{d\gamma}{dt} \right\| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \left| \frac{ds}{dt} \right| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \frac{ds}{dt}$.
 - But $s = \int \left\| \frac{d\gamma}{dt} \right\| dt \implies \frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\|$.
 - Compare last two lines. Then $\left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1$, so $\tilde{\gamma}(s)$ is unit speed.



Closed curves

Example:

- Ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, $p, q > 0$ are constants.
- Parametrize as $\gamma(t) = (p \cos t, q \sin t)$, $t \in \mathbb{R}$.
- Then $\gamma(t + 2\pi) = \gamma(t)$ for all $t \in \mathbb{R}$.
- γ is 2π -periodic.

Definition

- If $\gamma(t + T) = \gamma(t)$ for all t and for some $T > 0$, then γ is T -periodic.
- If $\gamma(t) = p$ for all t (where $p \in \mathbb{R}^n$ is a point), then γ is a *constant curve*.
- If γ is T -periodic and not constant, then γ is a *closed curve*.
- A *simple* closed curve has no self-intersections: $\gamma(t_1) \neq \gamma(t_2)$ whenever $|t_2 - t_1| < T$.

Jordan curve theorem

Simple closed curves, also called *Jordan curves*, are closed plane curves that do not self-intersect.

Theorem (Jordan curve theorem)

- *Every simple closed curve separates $\mathbb{R}^2$ into two disjoint regions, called the interior and exterior regions.*
 - *The interior region is bounded (contained within a circle).*
 - *The exterior region is unbounded.*
-
- Simple statement, surprisingly difficult to prove.
 - See graduate level algebraic topology texts for proof.

The isoperimetric inequality

Theorem

Let $\gamma : I \rightarrow \mathbb{R}^2$ be a simple closed curve of length $L(\gamma)$, enclosing a region of area $A(\gamma)$. Then

$$A(\gamma) \leq \frac{1}{4\pi} (L(\gamma))^2.$$

Equality holds iff γ is a circle.

This simple theorem has motivated a great many proofs and almost as many profound ideas. The most common proof uses

Theorem (Wirtinger's inequality)

Let $F : [0, \pi] \rightarrow \mathbb{R}$ be a smooth function with $F(0) = F(\pi) = 0$. Then

$$\int_0^\pi \left(\frac{dF}{dt} \right)^2 dt \geq \int_0^\pi (F(t))^2 dt,$$

and equality holds iff $F(t) = C \sin t$, $C = \text{const.}$

Proof of Wirtinger's inequality

Set-up:

- Define $G(t) = F(t)/\sin t$, $t \in (0, \pi)$.
- $\lim_{t \rightarrow 0^+} G(t) = \lim_{t \rightarrow 0^+} \frac{F(t)}{\sin t} = \lim_{t \rightarrow 0^+} \frac{F'(t)}{\cos t} = \lim_{t \rightarrow 0^+} F'(t)$. Exists because F is smooth. Likewise, $\lim_{t \rightarrow \pi^-} G(t)$ exists. So define $G(0)$, $G(\pi)$ by continuity (i.e., $G(0) := \lim_{t \rightarrow 0^+} G(t)$).
- Then $G : [0, \pi] \rightarrow \mathbb{R}$ is smooth.
- Then $F(t) = G(t) \sin t$, so $\dot{F}(t) = \dot{G}(t) \sin t + G(t) \cos t$.
- Use this and integration by parts to compute

$$\int_0^\pi \left(\dot{F}^2(t) - F^2(t) \right) dt = \int_0^\pi \dot{G}^2(t) \sin^2 t dt \geq 0.$$

- This proves the inequality.

Equality case

- Last slide: $\int_0^\pi \left(\dot{F}^2(t) - F^2(t) \right) dt = \int_0^\pi \dot{G}^2(t) \sin^2 t dt \geq 0.$
- From this, if $\int_0^\pi \left(\dot{F}^2(t) - F^2(t) \right) dt = 0$, then $\int_0^\pi \dot{G}^2(t) \sin^2 t dt = 0.$
- Because the integrand is nonnegative, the integral is zero only if $\dot{G}(t) \sin t = 0$ for all $t \in [0, \pi]$.
- Then $\dot{G}(t) = 0$, so $G(t) = C = \text{const.}$
- Since $G(t) = F(t)/\sin t$, we have $F(t) = C \sin t$, as required.

Proof of isoperimetric inequality

- Unit speed closed curve γ , arclength L , positioned so that $\gamma(0) = \mathbf{0}$.
- Reparametrize by $t = \frac{\pi s}{L}$. Then $t \in [0, \pi]$, speed is $\|\dot{\gamma}(t)\| = \frac{L}{\pi} = \text{const.}$
- Polar coordinates: $\gamma(t) = (r(t), \theta(t))$. Then

$$L^2 = \pi^2 \|\dot{\gamma}(t)\|^2 = \pi \int_0^\pi \|\dot{\gamma}(t)\|^2 dt = \pi \int_0^\pi (\dot{r}^2 + r^2 \dot{\theta}^2) dt. \quad (1)$$

- From Calculus, area enclosed by a polar curve is

$$A = \frac{1}{2} \int_0^\pi (x\dot{y} - \dot{x}y) dt = \frac{1}{2} \int_0^\pi r^2(t) \dot{\theta}(t) dt. \quad (2)$$

- Combine (1) and (2):

$$\frac{L^2}{4\pi} - A = \frac{1}{4} \int_0^\pi (\dot{r}^2 + r^2 \dot{\theta}^2 - 2r^2 \dot{\theta}) dt = \frac{1}{4} \int_0^\pi \left[\dot{r}^2 + r^2 (\dot{\theta}^2 - 2\dot{\theta}) \right] dt.$$

Isoperimetric inequality continued

- Complete the square:

$$\begin{aligned}\frac{L^2}{4\pi} - A &= \frac{1}{4} \int_0^\pi \left[\dot{r}^2 - r^2 + r^2 (\dot{\theta} - 1)^2 \right] dt \\ &\geq \frac{1}{4} \int_0^\pi [\dot{r}^2 - r^2] dt \\ &\geq 0\end{aligned}\tag{3}$$

by Wirtinger's inequality, which we recall says that $\int_0^\pi \dot{r}^2 dt \geq \int_0^\pi r^2 dt$ for any smooth function $r(t)$ such that $r(0) = r(\pi) = 0$.

- This proves the inequality.

Case of equality

We still have to show that $\frac{L^2}{4\pi} = A$ iff γ is a circle.

- If γ is a circle, then $L = 2\pi r$ so $\frac{L^2}{4\pi} = \pi r^2$.
- But if γ is a circle, then $A = \pi r^2$. Hence $\frac{L^2}{4\pi} = A$.
- Must prove converse: that if $\frac{L^2}{4\pi} = A$ then γ is a circle.
- Use $\frac{L^2}{4\pi} - A = 0$ in first line of (3):

$$\begin{aligned} 0 &= \frac{L^2}{4\pi} - A = \frac{1}{4} \int_0^\pi \left[\dot{r}^2 - r^2 + r^2 (\dot{\theta} - 1)^2 \right] dt \\ &= \frac{1}{4} \int_0^\pi [\dot{r}^2 - r^2] dt + \frac{1}{4} \int_0^\pi r^2 (\dot{\theta} - 1)^2 dt \end{aligned}$$

Equality case continued

- Last slide: $0 = \frac{1}{4} \int_0^\pi [\dot{r}^2 - r^2] dt + \frac{1}{4} \int_0^\pi r^2 (\dot{\theta} - 1)^2 dt.$
- By Wirtinger, first integral on right is nonnegative. Second integral on right is obviously nonnegative. Thus, each integral must vanish:

$$\int_0^\pi [\dot{r}^2 - r^2] dt = 0 \quad \text{and} \quad \int_0^\pi r^2 (\dot{\theta} - 1)^2 dt = 0.$$

- But $\int_0^\pi r^2 (\dot{\theta} - 1)^2 dt = 0 \implies \dot{\theta} = 1 \implies \theta = t + \theta_0$ for $\theta_0 = \text{const.}$

Simplify: Rotate axes to get $\theta_0 = 0$, then $\theta = t$.

- And $\frac{1}{4} \int_0^\pi [\dot{r}^2 - r^2] dt = 0 \implies r = C \sin t$ by the equality case of Wirtinger.
- So $r = C \sin \theta$, which is polar equation of *circle* that passes through the origin. (Exercise: Obtain the Cartesian form $x^2 + (y - \frac{C}{2})^2 = \frac{C^2}{4}$.)

Curvature

When is a curve ...*curved*?

Definition

If $\gamma : I \rightarrow \mathbb{R}^n$ is a unit speed curve, then its curvature is $\kappa := \|\ddot{\gamma}\|$.

Interpretation: Curvature as quadratic coefficient in Taylor's theorem:

$$\gamma(t_0 + \Delta t) = \gamma(t_0) + \dot{\gamma}(t_0)\Delta t + \frac{1}{2}\ddot{\gamma}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

- Can replace $\dot{\gamma}(t_0)$ by unit tangent $\mathbf{T}(t_0) = \dot{\gamma}(t_0)$.
- $\dot{\gamma}(t_0) \cdot \dot{\gamma}(t_0) = 1 \implies 2\dot{\gamma}(t_0) \cdot \ddot{\gamma}(t_0) = 0$, so $\ddot{\gamma} \perp \dot{\gamma}$ for a unit speed curve (if $\ddot{\gamma} \neq 0$).
- Then $\ddot{\gamma} = \pm\kappa\mathbf{N}$ where $\mathbf{N}$ is unit normal vector (orthogonal to $\mathbf{T}$).
- Get $\gamma(t_0 + \Delta t) = \gamma(t_0) + \mathbf{T}(t_0)\Delta t \pm \frac{1}{2}\kappa(t_0)\mathbf{N}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3)$
- Two choices for $\mathbf{N}$: we choose it so that $\{\mathbf{T}, \mathbf{N}\}$ is *right-handed*.

Example: Circle

- Circle in $\mathbb{R}^2$: $\gamma(t) = (x_0 + a \cos \frac{t}{a}, y_0 + a \sin \frac{t}{a})$, $t \in [0, 2\pi a)$.
- $\dot{\gamma} = (-\sin \frac{t}{a}, \cos \frac{t}{a})$, $\ddot{\gamma} = -\frac{1}{a} (\cos \frac{t}{a}, \sin \frac{t}{a})$.
- Then $\kappa = \frac{1}{a} \|(\cos \frac{t}{a}, \sin \frac{t}{a})\| = \frac{1}{a}$.
- Circles have constant curvature = 1/radius.

Osculating circles

Definition

If a curve $\gamma : I \rightarrow \mathbb{R}^2$ has curvature $\kappa(t) \neq 0$ at point $p = \gamma(t)$, we define its *radius of curvature* at p to be $\rho(t) = 1/\kappa(t)$.

The *osculating circle* to γ at p is the circle that

- passes through p ,
- has the same tangent line as γ at p ,
- has radius $\rho = \frac{1}{\kappa}$, and
- lies on the concave side of γ .

Signed curvature

- Parametrize the curve $\gamma(t)$ in $\mathbb{R}^2$.
- The direction of increasing parameter is the *orientation*.
- Define the unit tangent vector $\mathbf{T} = \dot{\gamma} / \|\dot{\gamma}\|$.
- Define the unit normal $\mathbf{N}$ by rotating $\mathbf{T}$ by $\frac{\pi}{2}$ *counter-clockwise* (also called the *right-handed sense*).
- Then the *signed curvature* κ_S is defined by

$$\ddot{\gamma}(s) = \kappa_S \mathbf{N}$$

where s is an arclength parameter with $ds/dt > 0$ (i.e., same orientation as t).

- Relation to (ordinary) curvature is $\kappa := |\kappa_S|$.

Interpretation: turning angle

Theorem (The turning angle)

There is a unique smooth function ϕ , called the turning angle, along the regular curve γ such that $\phi(s_0) = \phi_0$ and $\mathbf{T} = (\cos \phi(s), \sin \phi(s))$.

- Tangent vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:
 $\mathbf{T} = \dot{\gamma}(s) = (\cos \phi(s), \sin \phi(s))$
- Calculate: $\dot{\mathbf{T}} = \ddot{\gamma}(s) = \dot{\phi}(s) (-\sin \phi(s), \cos \phi(s))$
- Normal vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:

$$\mathbf{N} = \left(\cos \left(\phi(s) + \frac{\pi}{2} \right), \sin \left(\phi(s) + \frac{\pi}{2} \right) \right) = (-\sin \phi(s), \cos \phi(s))$$

- Conclude that $\ddot{\gamma}(s) = \dot{\phi}(s)\mathbf{N}$.
- Compare to $\ddot{\gamma}(s) = \kappa_S \mathbf{N}$ to get $\kappa_S(s) = \dot{\phi}(s)$.
- The signed curvature is the rate of change of the turning angle with respect to arclength.

Hopf's Umlaufsatz (rotation rate)

- Integrate $\kappa_S(s) = \dot{\phi}(s)$ over curve γ .
- $\int_{s_0}^s \kappa_S(u) du = \int_{s_0}^s \dot{\phi}(u) du = \phi(s) - \phi(s_0)$.
- Take γ closed, with period T .
- $\int_{s_0}^{s_0+T} \kappa_S(u) du = \phi(s_0 + T) - \phi(s_0)$.
- But $\phi(s_0 + T) - \phi(s_0) = 2\pi k$, $k \in \mathbb{Z}$.
- In fact, can argue that $k = \pm 1$ if curve traversed once; k is the *winding number*.

Theorem (Hopf's Umlaufsatz)

The total curvature of a closed curve of period T is $\int_{s_0}^{s_0+T} \kappa_S(u) du = \pm 2\pi$.

A useful formula

- $\mathbf{T} = \frac{\partial \gamma}{\partial s} = \text{unit tangent vector.}$
- Last slide: $\frac{\partial^2 \gamma}{\partial s^2} = \kappa_S \mathbf{N}.$
- Then $\frac{\partial^3 \gamma}{\partial s^3} = \kappa_S \frac{\partial \mathbf{N}}{\partial s} + \frac{\partial \kappa_S}{\partial s} \mathbf{N}.$
- And then $\frac{\partial \gamma}{\partial s} \cdot \frac{\partial^3 \gamma}{\partial s^3} = \kappa_S \mathbf{T} \cdot \frac{\partial \mathbf{N}}{\partial s} + \frac{\partial \kappa_S}{\partial s} \mathbf{T} \cdot \mathbf{N} = \kappa_S \mathbf{T} \cdot \frac{\partial \mathbf{N}}{\partial s} = \kappa_S \frac{\partial \gamma}{\partial s} \cdot \frac{\partial \mathbf{N}}{\partial s}.$
- But $\frac{\partial \gamma}{\partial s} \cdot \frac{\partial^3 \gamma}{\partial s^3} = \frac{\partial}{\partial s} \left(\frac{\partial \gamma}{\partial s} \cdot \frac{\partial^2 \gamma}{\partial s^2} \right) - \frac{\partial^2 \gamma}{\partial s^2} \cdot \frac{\partial^2 \gamma}{\partial s^2} = -\frac{\partial^2 \gamma}{\partial s^2} \cdot \frac{\partial^2 \gamma}{\partial s^2} = -\kappa_S^2.$
- Comparing the last two lines, then

$$\frac{\partial \gamma}{\partial s} \cdot \frac{\partial \mathbf{N}}{\partial s} = -\kappa_S.$$

Application: Curve shortening flow (CSF)

- From here on, I will write κ when I mean κ_S .
- Let $\gamma(u)$ be a parametrized curve. Let $\mathbf{N}(u)$ be the normal vector to the curve. Let $\gamma(t, s)$ be a smooth family of such curves, one for each value of $t \in (-T, T)$ for some $T \in (0, \infty]$. The curve shortening flow is

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{N}(t).$$

- A more useful definition uses the arclength parametrization and the inner product $\langle \cdot, \cdot \rangle$ (dot product on $\mathbb{R}^2$):

$$\left\langle \mathbf{N}, \frac{\partial \gamma}{\partial t} \right\rangle = \kappa.$$

- Online demonstration at <https://a.carapetis.com/csf/>

A simple curve shortening flow: Circle

- $\gamma(s) = a \left(\cos \frac{s}{a}, \sin \frac{s}{a} \right)$ (we'll let $a = a(t)$).
- $\gamma'(s) = \mathbf{T} = \left(-\sin \frac{s}{a}, \cos \frac{s}{a} \right)$.
- $\gamma''(s) = \kappa \mathbf{N} = -\frac{1}{a} \left(\cos \frac{s}{a}, \sin \frac{s}{a} \right)$.
- $\dot{\gamma} = \frac{\partial \gamma}{\partial t} = \dot{a} \left(\cos \frac{s}{a}, \sin \frac{s}{a} \right) - \frac{s\dot{a}}{a} \left(-\sin \frac{s}{a}, \cos \frac{s}{a} \right)$.
- $\langle \mathbf{N}, \dot{\gamma} \rangle = -\dot{a}$.
- But by CSF, $\langle \mathbf{N}, \dot{\gamma} \rangle = \kappa = \frac{1}{a}$.
- Therefore $-\dot{a} = \frac{1}{a}$, so $2a\dot{a} \equiv \frac{\partial}{\partial t} (a^2) = -2$.
- If the initial radius is $a(0) = a_0$, then we get $a^2(t) = a_0^2 - 2t$.
- The flow is $\gamma(s, t) = \sqrt{a_0^2 - 2t} \left(\cos \frac{s}{\sqrt{a_0^2 - 2t}}, \sin \frac{s}{\sqrt{a_0^2 - 2t}} \right)$
- The circle disappears at time $t = \frac{a_0^2}{2}$.

A CSF soliton: the “grim reaper”

- $\gamma(u, t) = (u, t - \log \cos u)$.
- $\gamma' = (1, \tan u)$, so $\frac{ds}{du} = \|\gamma'\| = \sec u$.
- $\mathbf{T} = \frac{(1, \tan u)}{\sqrt{1 + \tan^2 u}} = (\cos u, \sin u)$.
- $\mathbf{N} = (-\sin u, \cos u)$.
- $\dot{\gamma} = (0, 1)$ (velocity is vertical and constant).
- $\langle \mathbf{N}, \dot{\gamma} \rangle = \cos u$.
- So this is a CSF if $\kappa = \cos u$.
- By direct computation, we have $\kappa \mathbf{N} = \frac{\partial}{\partial s} \mathbf{T} = \frac{\partial u}{\partial s} \frac{\partial \mathbf{T}}{\partial u} = \cos u (-\sin u, \cos u)$
so indeed $\kappa = \cos u$!

How to prove the isoperimetric inequality with CSF

Say that for each $t \in [0, T)$, $\gamma(u, t)$ is a closed curve with arclength function $s = s(u, t)$ and say that $\frac{\partial \gamma}{\partial t} = \kappa \mathbf{N}$. Four claims:

- ❶ The length $L(t)$ of the curve obeys $\frac{dL}{dt} = - \int \kappa^2 ds$.
- ❷ The enclosed area $A(t)$ obeys $\frac{dA}{dt} = - \int \kappa ds$.
 - Then $\frac{dA}{dt} = -2\pi$ by Hopf's Umlaufsatz.
- ❸ Any such flow ends in finite time, say T .
- ❹ As $t \nearrow T$, the curvature approaches a uniform function of time:
 $\kappa(s, t) \nearrow k(t) \rightarrow \infty$.
 - Then the curve becomes a circle of radius $\frac{1}{k(t)}$.

Sketch of proof

If the Claims are true, then

$$\begin{aligned}\frac{d}{dt} (L^2 - 4\pi A) &= 2L \frac{dL}{dt} - 4\pi \frac{dA}{dt} \\ &= -2L \int_{\gamma} \kappa^2 ds + 4\pi \int_{\gamma} \kappa ds \\ &\leq -2 \left(\int_{\gamma} \kappa ds \right)^2 + 4\pi \int_{\gamma} \kappa ds \quad (\text{Hölder inequality}) \\ &= -2 \left(\int_{\gamma} \kappa ds \right)^2 + 2 \left(\int_{\gamma} \kappa ds \right)^2 \quad (\text{Umlaufsatz}) \\ &= 0.\end{aligned}$$

and since $L^2(t) - 4\pi A(t) \rightarrow 0$ as $t \nearrow T$ by Claim 3, then at every $0 \leq t < T$ we have $L^2 - 4\pi A \geq 0$. So $L^2 - 4\pi A \geq 0$ for the initial curve $\gamma(u, 0)$.

- This is the isoperimetric inequality. Now we must prove the Claims, which are facts about CSF.

Claim 1: $\frac{dL}{dt} = - \int \kappa^2 ds$

- If u is a parameter for the curve $\gamma(u) = (x(u), y(u))$ and s is an arclength parameter then $ds = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du = \|\gamma'(u)\| du$. Then

$$\begin{aligned}\frac{\partial}{\partial t} \left(\left\| \frac{\partial \gamma}{\partial u} \right\|^2 \right) &= 2 \frac{\partial \gamma}{\partial u} \cdot \frac{\partial}{\partial u} \frac{\partial \gamma}{\partial t} \\ &= 2 \frac{\partial \gamma}{\partial u} \cdot \frac{\partial}{\partial u} (\kappa \mathbf{N}) \text{ by the CSF equation} \\ &= 2 \left\| \frac{\partial \gamma}{\partial u} \right\|^2 \frac{\partial \gamma}{\partial s} \cdot \frac{\partial}{\partial s} (\kappa \mathbf{N}) \text{ since } \frac{\partial}{\partial u} = \left\| \frac{\partial \gamma}{\partial u} \right\| \frac{\partial}{\partial s} \\ &= -2\kappa^2 \left\| \frac{\partial \gamma}{\partial u} \right\|^2,\end{aligned}$$

using that $\frac{\partial \gamma}{\partial s} \cdot \frac{\partial}{\partial s} (\kappa \mathbf{N}) = \mathbf{T} \cdot \left(\frac{\partial \kappa}{\partial s} \mathbf{N} + \kappa \frac{\partial \mathbf{N}}{\partial s} \right) = -\kappa^2$ since $\mathbf{T} \cdot \mathbf{N} = 0$ and $\mathbf{T} \cdot \frac{\partial \mathbf{N}}{\partial s} = -\kappa$ by our earlier “useful equation”.

Claim 1 continued: $\frac{dL}{dt} = - \int \kappa^2 ds$

Then

$$\frac{\partial}{\partial t} \left(\left\| \frac{\partial \gamma}{\partial u} \right\| \right) = -\kappa^2 \left\| \frac{\partial \gamma}{\partial u} \right\|,$$

and so

$$\begin{aligned} \frac{dL}{dt} &= \frac{d}{dt} \int ds \\ &= \frac{d}{dt} \int \left\| \frac{\partial \gamma}{\partial u} \right\| du \\ &= \int \frac{\partial}{\partial t} \left\| \frac{\partial \gamma}{\partial u} \right\| du \\ &= - \int \kappa^2 \left\| \frac{\partial \gamma}{\partial u} \right\| du \\ &= - \int \kappa^2 ds. \end{aligned}$$

QED

Claim 2: $\frac{dA}{dt} = - \int \kappa ds$

- $A = \frac{1}{2} \int_R (x dy - y dx) = \frac{1}{2} \int_\gamma (x, y) \cdot \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s} \right) ds = -\frac{1}{2} \int_\gamma \gamma \cdot \mathbf{N} ds.$
- Then $\frac{dA}{dt} = -\frac{1}{2} \int_\gamma \left(\frac{\partial \gamma}{\partial t} \cdot \mathbf{N} + \gamma \cdot \frac{\partial \mathbf{N}}{\partial t} \right) ds + \frac{1}{2} \int \gamma \cdot \mathbf{N} \kappa^2 ds$, where the κ^2 arises in the same way it did when proving Claim 1.
- Now $\frac{\partial \gamma}{\partial t} \cdot \mathbf{N} = \kappa$ from the CSF equation, so $\frac{\partial}{\partial s} \left(\frac{\partial \gamma}{\partial t} \cdot \mathbf{N} \right) = \frac{\partial \kappa}{\partial s}.$
- Can show that $\frac{\partial \mathbf{N}}{\partial t} = -\frac{\partial \kappa}{\partial s} \mathbf{T}$ by differentiating the flow equation.

$$\begin{aligned}
 \implies \frac{dA}{dt} &= \frac{1}{2} \int_\gamma \left(-\kappa + \frac{\partial \kappa}{\partial s} \gamma \cdot \mathbf{T} + \gamma \cdot \mathbf{N} \kappa^2 \right) ds \\
 &= \frac{1}{2} \int_\gamma \left(-\kappa - \kappa \frac{\partial \gamma}{\partial s} \cdot \mathbf{T} - \kappa \gamma \cdot \frac{\partial \mathbf{T}}{\partial s} + \gamma \cdot \mathbf{N} \kappa^2 \right) ds \\
 &= \frac{1}{2} \int_\gamma \left(-2\kappa - \kappa^2 \gamma \cdot \mathbf{N} + \gamma \cdot \mathbf{N} \kappa^2 \right) ds \text{ using } \frac{\partial^2 \gamma}{\partial s^2} = \frac{\partial \mathbf{T}}{\partial s} = \kappa \mathbf{N} \\
 &= - \int_\gamma \kappa ds \quad \text{QED}
 \end{aligned}$$

Claims 3 and 4

- Sketch of Claim 3: Closed curves vanish in finite time
 - Every close curve γ is bounded.
 - Draw a circle enclosing γ .
 - The circle vanishes in finite time T under CSF (we saw that earlier).
 - The flowing closed curve remains always inside the flowing circle (why?).
 - Hence the flowing closed curve must vanish at some time $t \leq T$,
- We will omit the proof of Claim 4, which requires more background.
- For details omitted here, see the references on the next slide.

Conclusion

References:

- ME Gage, *An isoperimetric inequality with applications to curve-shortening*, Duke Math J 50 (1983) 1225–1229.
- MA Grayson, *The heat equation shrinks embedded plane curves to round points*, Journal of Differential Geometry 26(2) (1987) 285–314.
- B Andrews, B Chow, C Guenther, and M Langford, *Extrinsic geometric flows*, Graduate Studies in Mathematics 206 (American Mathematical Society, Providence, 2020).

There are many other problems that can be treated using curvature flows.

- Can the isoperimetric inequality on the sphere be proved this way?