Project for ergodic theory: weighted Birkhoff ergodic theorem

1. Background.

The Birkhoff Ergodic Theorem states that for an ergodic measure-preserving dynamical system $(X, \mathcal{B}, \mu, T)$ and an integrable observable $f \in L^1(\mu)$,

$$B_N(f,x) := \frac{1}{N}\sum_{n=0}^{N-1} f(T^n x) \longrightarrow \int_X f\,d\mu \quad \text{for } \mu\text{-a.e. } x \in X\,. \tag{1}$$

Although Birkhoff ergodic theorem guarantees the convergence, it does not provide a general *convergence rate*. In fact, the convergence, in general, is quite slow: for non-trivial (non-identically constant) observables, the convergence rate is at most $O(1/N)$. This slow convergence is often unsatisfactory for practical numerical computations.

To address this issue, the *weighted Birkhoff average* is introduced

$$WB_N(f,x) := \frac{1}{\alpha_N}\sum_{n=0}^{N-1} w(n/N) f(T^n x)$$

where $\alpha_N = \sum_{n=0}^{N-1} w(n/N)$. A common choice of the weight function is a smooth 'bump' function whose derivatives vanish at the boundaries, i.e., functions from the class

$$\mathcal{W} = \left\{ w \in C^\infty([0,1], \mathbb{R}_{\geq 0}) : \int_0^1 w(x)dx = 1,\ \frac{d^i w}{dx^i}(0) = \frac{d^i w}{dx^i}(1) = 0,\ \forall i = 0,1,2,\ldots \right\}.$$

A standard example is

$$w(x) = \begin{cases} Ce^{-1/x(1-x)}, & x \in (0,1), \\ 0, & x = 0,1. \end{cases}$$

where C is chosen to satisfy $\int_0^1 w(x)dx = 1$.

The *weighted Birkhoff theorem* shows that under appropriate assumptions on the weights, $WB_N(f,x)$ converge to the same limit as $B_N(f,x)$, i.e.,

$$WB_N(f,x) \longrightarrow \int_X f\,d\mu, \qquad \text{for } \mu\text{-a.e. } x \in X\,. \tag{2}$$

This extends the classical Birkhoff theorem, but the convergence rates in (1) and (2) can be very different. Recent research has shown that the convergence rate may be greatly improved beyond the unweighted $O(1/N)$ rate:

(1) If T is periodic (e.g., rational rotation on a circle), the rate of convergence in (2) is exponential.

(2) If T is quasi-periodic (i.e., irrational rotation on a circle) and $f \in C^\infty$, the rate is $O(N^{-m})$ for any $m > 0$; if f is analytic, it again becomes exponential.

(3) For chaotic dynamics f (e.g., expanding maps (the doubling map) on interval), no general analytic rate is known, but empirical evidence suggests that the weighted and unweighted averages share the same rate $O(1/N)$.

These developments have made weighted Birkhoff averages an effective numerical tool for computing dynamical quantities with high accuracy.

2. Project Questions.

(You may choose either of the following questions, or both if you wish.)

Q1. Investigate whether weighted Birkhoff averages can improve convergence rates under different assumptions on the dynamics. In particular, periodic systems, quasi-periodic systems, and chaotic systems.

- Summarize the known convergence rates from the literature and explain why they differ.
- What is the possible mechanisms for the weighting accelerate convergence?
- Why can smooth weights eliminate much of this error for quasi-periodic orbits? Why does the same mechanism fail for chaotic systems?

Support your discussion with theoretical arguments from the literature and numerical examples.

Q2. Weighted Birkhoff ergodic theorem in applications.

Survey the literature on weighted Birkhoff averages and describe at least two applications of the method. Possible applications include:

- Computing invariant measures,
- Computing rotation numbers, computing Lyapunov exponents,
- Identifying periodic, quasi-periodic, and chaotic trajectories,
- Computing invariant curves or invariant tori.
- ...

For each application, explain the mathematical problem, why the classical Birkhoff average is insufficient or inefficient, how weighted averaging improves the computation.