# IUSEP: Isoperimetric inequality

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# Elementary Calculus

- A farmer wishes to enclose a rectangular region $R$ with a fence.
- The fence has total length $L = const > 0$.
- Show that the area $A$ that can be enclosed obeys

$$A \leq \frac{L^2}{16}$$

and that

$$A = \frac{L^2}{16}$$

if and only if $R$ is a square.

This simple problem has a serious flaw: why should the region $R$ be a rectangle?

- A circle of perimeter $L$ encloses area $A = \pi \cdot \left(\frac{L}{2\pi}\right)^2 = \frac{L^2}{4\pi} > \frac{L^2}{16}$.

# The isoperimetric problem

1. In the plane:
   - Let $I \subset \mathbb{R}$ be a connected interval of the real line.
   - Let $\gamma : I \to \mathbb{R}$ be a simple closed curve of arclength $L$, enclosing a region $R$ of area $A$.
   - Show that
   $$L^2 - 4\pi A \geq 0$$
   and that $L^2 = 4\pi A$ if and only if $\gamma$ is a circle.
2. In the 2-sphere of radius $a$:
   - Show that
   $$L^2 - 4\pi A + \frac{A^2}{a^2} \geq 0$$
   and that $L^2 = 4\pi A$ if and only if $\gamma$ is a *great circle*.

We will prove the first of these two statements, but first we have to define everything. The key tools are *differential geometry* and *geometric analysis*.

# Parametrized curves

**Definition**

A *parametrized curve* is a map $\gamma : I \to \mathbb{R}^n$, where $I$ is a connected interval of $\mathbb{R}$.

- Example: It's very easy to parametrize a graph $y = f(x)$.
  - Just choose $x$ to *be* the parameter; i.e., write

$$x(t) = t,$$
$$y(t) = f(t).$$

- Don't forget to choose domain; for example, perhaps

$$t \in I$$

for some interval $I \subset \mathbb{R}$.

# More examples

- The parametrized curve $\begin{cases} x(t) = & t \ , \\ y(t) = & \sqrt{a^2 - t^2} \ , \\ t \in & [-a, a] \ , \end{cases}$ is a semi-circle.
- The parametrized curve $\begin{cases} x(t) = & \cos t \ , \\ y(t) = & \sin t \ , \\ t \in & [0, 2\pi) \ , \end{cases}$ is a circle, traversed once counter-clockwise.
- The parametrized curve $\begin{cases} x(t) = & \cos t \ , \\ y(t) = & \sin t \ , \\ t \in & [0, 4\pi) \ , \end{cases}$ is a circle, traversed twice counter-clockwise.

Notice the parametrization carries extra information not available from the graphical description of a curve.

# Example: The astroid

- The parametrized curve $\gamma(t) = \left(\cos^3 t, \sin^3 t\right)$, $t \in [0, 2\pi)$, is called an *astroid*.
- Can write it as $\begin{cases} x(t) = \cos^3 t \\ y(t) = sin^3 t \\ t \in [0, 2\pi) \end{cases}$
- Then $x^{2/3} = \cos^2 t$ and $y^{2/3} = \sin^2 t$, so $x^{2/3} + y^{2/3} = 1$.
- Graphical form: $y = \pm \left(1 - x^{2/3}\right)^{3/2}$.
- Level set form:
  - Let $z = f(x, y) = x^{2/3} + y^{2/3}$.
  - Then the astroid is the level set $z = f(x, y) = 1$.
- Graphical and level set forms have less information than parametrized form.

# Tangent vectors

- Recall tangent line to graph $y = f(x)$ at $(x_0, y_0)$ is $y - y_0 = f'(x_0)\,(x - x_0)$.
- Tangent vector: Any (non-zero) vector parallel to tangent line.
- Parametrized form of line: Take $s \in \mathbb{R}$ and
  $x(s) = x_0 + s$
  $y(s) = y_0 + f'(x_0)s$
- Differentiate wrt $s$: $x'(s) = 1$, $y'(s) = f'(x_0)$.
- Tangent vectors to line are the vectors parallel to $(1, f'(x_0))$.

# Tangent vectors to parametrized curves

- Parametrized curve $\gamma : I \to \mathbb{R}^n$ is a *vector-valued* function.
- $\gamma(t) = (\gamma_1(t), \gamma_x(t), \ldots, \gamma_n(t)) = (x_1(t), x_2(t), \ldots, x_n(t))$.

Definition

$$\gamma'(t) = \dot{\gamma}(t) = \frac{d\gamma}{dt} = \left( \frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt} \right)$$
$$= \lim_{\Delta t \to 0} \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t}$$

Then $\gamma'(t)$ is a *tangent vector* to curve $\gamma$ at $t$ provided $\gamma'(t) \neq (0, \ldots, 0)$.

(Generally, we will just write 0 even if we mean the 0-vector $(0, 0, \ldots, 0)$.)

# Example

- $\gamma(t) = t^3\mathbf{e}_1 + t^2\mathbf{e}_2,\ t \in \mathbb{R}$,
  $\{\mathbf{e}_1, \mathbf{e}_2\}$ = orthonormal basis (ONB).
- $\left.\begin{aligned} x(t) &= t^3 \\ y(t) &= t^2 \end{aligned}\right\} \implies y = x^{2/3}$.
- Chain rule:
  - $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \implies 2t = \frac{dy}{dx} \cdot 3t^2 \implies \frac{dy}{dx} = \frac{2t}{3t^2}$ undefined at $t = 0$.

**Definition**

A parametrized curve $\gamma : I \to \mathbb{R}^n$ is

- *smooth* at $t_0 \in I$ if all derivatives of all components $x_i(t)$ exist at $t = t_0$, and
- *regular* at $t_0 \in I$ if it is smooth at $t_0$ and $\frac{d\gamma}{dt}(t_0) \neq (0, \dots, 0)$; otherwise $t_0$ is a *singular point*.

The above example is smooth but not regular at $t = 0$.

# Arclength

- Recall arclength in $\mathbb{R}^2$:
  $s = \int ds = \int \sqrt{dx^2 + dy^2} = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
- In $\mathbb{R}^n$: $s = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \cdots + \left(\frac{dx_n}{dt}\right)^2} dt = \int_{t_0}^{t_1} \sqrt{\frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt}} dt = \int_{t_0}^{t_1} \left\| \frac{d\gamma}{dt} \right\| dt$

## Definition

The arclength function of a curve $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is
$s := \int_{t_0}^{t} \left\| \frac{d\gamma(t')}{dt'} \right\| dt'$
for $t \in [t_0, t_1]$.

Fundamental Theorem of Calculus $\implies \frac{ds}{dt} = \left\| \frac{d\gamma(t)}{dt} \right\|$.
This is called the *speed* of the curve.

# Example: Log spiral

- The logarithmic spiral is the curve $\gamma(t) = e^t (\cos t, \sin t)$.
- $\gamma'(t) = e^t (\cos t - \sin t, \sin t + \cos t)$
- $\|\gamma'\| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2} e^t$.
- $s(t) = \int_{t_0}^{t} \sqrt{2} e^\tau \, d\tau = \sqrt{2} (e^t - e^{t_0})$.
- $t_0 \to -\infty \implies \gamma(t_0) \to (0,0),\ s(t) \to \sqrt{2} e^t$.
- $\gamma : (-\infty, t] \to \mathbb{R}^2$ has finite arclength, but no initial endpoint.

# Unit speed curves

- If $\|\dot{\gamma}(t)\| = 1$, $\gamma$ is *unit speed* and $t$ is an *arclength parameter* or *unit speed parameter*.
- If $\|\dot{\gamma}(t)\| = k = const > 0$, $\gamma$ is *constant speed* and $t$ is an *affine parameter*.
- Fact:
  - Let $\mathbf{v}$ be any unit vector field $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1$.
  - Let $\gamma(t)$ be a unit speed curve.
  - $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d}{dt}(1) = 0$.
  - But then $\frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = 0$.
  - Chain rule: $\dot{\gamma} \cdot \ddot{\gamma} = 0$.
  - Conclude that $\dot{\gamma} \perp \ddot{\gamma}$ along any unit speed curve whenever acceleration $\ddot{\gamma} \neq 0$.
  - For unit speed curves, write $\mathbf{t} := \dot{\gamma} =$ unit tangent vector. Note that $\|\mathbf{t}\| = \sqrt{\mathbf{t} \cdot \mathbf{t}} = 1$.

# Reparametrization

- Say $\gamma : (a, b) \to \mathbb{R}^n$ is a curve, and
- Say $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^n$ is a curve.

### Definition

If

- there is a smooth map $\phi : (\tilde{a}, \tilde{b}) \to (a, b)$
- with smooth inverse $\phi^{-1} : (a, b) \to (\tilde{a}, \tilde{b})$, such that
- $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) = (\gamma \circ \phi)(\tilde{t}) = \gamma(t)$ for all $\tilde{t} \in \left(\tilde{a}, \tilde{b}\right)$,

then $\tilde{\gamma} = \gamma \circ \phi$ is a *reparametrization* of $\gamma$.

# Theorem

## Theorem

*Any reparametrization of a regular curve is also a regular curve.*

## Proof.

- Let $t = \phi(\tilde{t})$ and $\tilde{\gamma}(\tilde{t}) = \gamma(t)$.
- Then $\tilde{t} = \phi^{-1}(t)$ so $t = \phi(\tilde{t}) = \phi(\phi^{-1}(t))$.
- Chain rule: $\frac{d\phi}{d\tilde{t}} \frac{d(\phi^{-1})}{dt} = 1$, so $\frac{d\phi}{d\tilde{t}} \neq 0$.
- $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d}{d\tilde{t}}(\gamma(t)) = \frac{d\gamma}{dt}\frac{d\phi}{d\tilde{t}}$.
- Now $\gamma$ is regular so $\frac{d\gamma}{dt} \neq 0$, and $\frac{d\phi}{d\tilde{t}} \neq 0$.
- Thus $\frac{d\tilde{\gamma}}{d\tilde{t}} \neq 0$.

$\square$

Works because the reparametrization $\phi$ is smooth with smooth inverse.

# The arclength function of a regular curve is smooth

- Say $\gamma : I \to \mathbb{R}^2 : t \to (x(t), y(t))$ is a regular curve.
- Then $x(t)$ and $y(t)$ are smooth functions.
- The square root function $f(w) = \sqrt{w}$ is smooth if $w \neq 0$.
- Since $\gamma$ is regular, $\dot{x}^2 + \dot{y}^2 \neq 0$.
- Thus $\frac{ds}{dt}(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$ is smooth.
- Therefore $s(t) = \int\limits_{t_0}^{t} \frac{ds}{dt'}(t')dt'$ is smooth.

# Regular curves have unit speed parametrizations

**Theorem**

*A parametrized curve has an arclength parametrization iff it is regular.*

**Proof.**

- Curve $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^2$ and reparametrization $t = \phi(\tilde{t})$, such that $\gamma(t) = \tilde{\gamma}(\tilde{t})$.
- Chain rule: $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt}\frac{dt}{d\tilde{t}} \implies \left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = \left\| \frac{d\gamma}{dt} \right\| \left| \frac{dt}{d\tilde{t}} \right|$.

$\Rightarrow$ If $\tilde{t}$ is arclength, then $\left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = 1$, so $\frac{d\gamma}{dt}$ is never zero. Then $\gamma(t)$ is regular.

$\Leftarrow$
- If $\frac{d\gamma}{dt} \neq 0$, then $\frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\| \neq 0$, so $s$ is smooth and strictly increasing.
- Then $\frac{d\gamma}{dt} = \frac{d\tilde{\gamma}}{ds}\frac{ds}{dt} \implies \left\| \frac{d\gamma}{dt} \right\| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \left| \frac{ds}{dt} \right| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \frac{ds}{dt}$.
- But $s = \int \left\| \frac{d\gamma}{dt} \right\| dt \implies \frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\|$.
- Compare last two lines. Then $\left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1$, so $\tilde{\gamma}(s)$ is unit speed.

□

# Closed curves

Example:

- Ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, $p, q > 0$ are constants.
- Parametrize as $\gamma(t) = (p \cos t, q \sin t)$, $t \in \mathbb{R}$.
- Then $\gamma(t + 2\pi) = \gamma(t)$ for all $t \in \mathbb{R}$.
- $\gamma$ is $2\pi$-periodic.

Definition

- If $\gamma(t + T) = \gamma(t)$ for all $t$ and for some $T > 0$, then $\gamma$ is *$T$-periodic*.
- If $\gamma(t) = p$ for all $t$ (where $p \in \mathbb{R}^n$ is a point), then $\gamma$ is a *constant curve*.
- If $\gamma$ is $T$-periodic and not constant, then $\gamma$ is a *closed curve*.
- A *simple* closed curve has no self-intersections: $\gamma(t_1) \neq \gamma(t_2)$ whenever $|t_2 - t_1| < T$.

# Jordan curve theorem

Simple closed curves, also called *Jordan curves*, are closed plane curves that do not self-intersect.

**Theorem (Jordan curve theorem)**

*Every simple closed curve separates* $\mathbb{R}^2$ *into two disjoint regions, called the interior and exterior regions.*
*The interior region is bounded (contained within a circle).*
*The exterior region is unbounded.*

Simple statement, surprisingly difficult to prove:
see graduate level algebraic topology texts for proof.

# The isoperimetric inequality

Theorem

*Let* $\gamma : I \to \mathbb{R}^2$ *be a simple closed curve of length* $L(\gamma)$*, enclosing a region of area* $A(\gamma)$*. Then*

$$A(\gamma) \leq \frac{1}{4\pi} (L(\gamma))^2 .$$

*Equality holds iff* $\gamma$ *is a circle.*

This simple theorem has motivated a great many proofs and almost as many profound ideas. The most common proof uses

Theorem (Wirtinger's inequality)

*Let* $F : [0, \pi] \to \mathbb{R}$ *be a smooth function with* $F(0) = F(\pi) = 0$*. Then*

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi (F(t))^2 \, dt,$$

*and equality holds iff* $F(t) = C \sin t$*,* $C = const$*.*

# Proof of Wirtinger's inequality

Set-up:

- Define $G(t) = F(t)/\sin t$, $t \in (0, \pi)$.
- $\lim_{t\to 0^+} G(t) = \lim_{t\to 0^+} \frac{F(t)}{\sin t} = \lim_{t\to 0^+} \frac{F'(t)}{\cos t} = \lim_{t\to 0^+} F'(t)$. Exists because $F$ is smooth. Likewise, $\lim_{t\to\pi^-} G(t)$ exists. So define $G(0)$, $G(\pi)$ by continuity (i.e., $G(0) := \lim_{t\to 0^+} G(t)$).
- Then $G : [0, \pi] \to \mathbb{R}$ is smooth.
- Then $F(t) = G(t)\sin t$, so $\dot{F}(t) = \dot{G}(t)\sin t + G(t)\cos t$.
- Use this and integration by parts to compute

$$\int_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = \int_0^\pi \dot{G}^2(t)\sin^2 t dt \geq 0.$$

- This proves the inequality.

# Equality case

- Last slide: $\int_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = \int_0^\pi \dot{G}^2(t) \sin^2 t dt \geq 0.$
- From this, if $\int_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = 0$, then necessarily $\int_0^\pi \dot{G}^2(t) \sin^2 t dt = 0.$
- Because the integrand is nonnegative, the integral is zero only if $\dot{G}(t) \sin t = 0$ for all $t \in [0, \pi]$.
- Then $\dot{G}(t) = 0$, so $G(t) = C = const$.
- Since $G(t) = F(t)/\sin t$, we have $F(t) = C \sin t$, as required.

# Proof of isoperimetric inequality

- Unit speed closed curve $\gamma$, arclength $L$, positioned so that $\gamma(0) = \mathbf{0}$.
- Reparametrize by $t = \frac{\pi s}{L}$. Then $t \in [0, \pi]$, speed is $\|\dot{\gamma}(t)\| = \frac{L}{\pi} = const$.
- Polar coordinates: $\gamma(t) = (r(t), \theta(t))$. Then

$$L^2 = \pi^2 \|\dot{\gamma}(t)\|^2 = \pi \int_0^\pi \|\dot{\gamma}(t)\|^2 \, dt = \pi \int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2\right) dt. \qquad (1)$$

- From Calculus, area enclosed by a polar curve is

$$A = \frac{1}{2} \int_0^\pi (x\dot{y} - \dot{x}y) \, dt = \frac{1}{2} \int_0^\pi r^2(t)\dot{\theta}(t) dt. \qquad (2)$$

- Combine (1) and (2):

$$\frac{L^2}{4\pi} - A = \frac{1}{4} \int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta}\right) dt = \frac{1}{4} \int_0^\pi \left[\dot{r}^2 + r^2 \left(\dot{\theta}^2 - 2\dot{\theta}\right)\right] dt.$$

# Isoperimetric inequality continued

- Complete the square:

$$\begin{aligned} \frac{L^2}{4\pi} - A &= \frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2 + r^2\left(\dot{\theta} - 1\right)^2\right] dt \\ &\geq \frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2\right] dt \\ &\geq 0 \end{aligned} \tag{3}$$

  by Wirtinger's inequality, which we recall says that $\int_0^\pi \dot{r}^2 dt \geq \int_0^\pi r^2 dt$ for any smooth function $r(t)$ such that $r(0) = r(\pi) = 0$.

- This proves the inequality.

# Case of equality

We still have to show that $\frac{L^2}{4\pi} = A$ iff $\gamma$ is a circle.

- If $\gamma$ is a circle, then $L = 2\pi r$ so $\frac{L^2}{4\pi} = \pi r^2$.
- But if $\gamma$ is a circle, then $A = \pi r^2$. Hence $\frac{L^2}{4\pi} = A$.
- Must prove converse: that if $\frac{L^2}{4\pi} = A$ then $\gamma$ is a circle.
- Use $\frac{L^2}{4\pi} - A = 0$ in first line of (3):

$$
\begin{aligned}
0 = \frac{L^2}{4\pi} - A &= \frac{1}{4}\int_0^{\pi} \left[\dot{r}^2 - r^2 + r^2\left(\dot{\theta} - 1\right)^2\right] dt \\
&= \frac{1}{4}\int_0^{\pi} \left[\dot{r}^2 - r^2\right] dt + \frac{1}{4}\int_0^{\pi} r^2\left(\dot{\theta} - 1\right)^2 dt
\end{aligned}
$$

# Equality case continued

- Last slide: $0 = \frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2\right] dt + \frac{1}{4}\int_0^\pi r^2 \left(\dot{\theta} - 1\right)^2 dt$.
- By Wirtinger, first integral on right is nonnegative. Second integral on right is obviously nonnegative. Thus, each integral must vanish:

$$\int_0^\pi \left[\dot{r}^2 - r^2\right] dt = 0 \quad \text{and} \quad \int_0^\pi r^2 \left(\dot{\theta} - 1\right)^2 dt = 0.$$

- But $\int_0^\pi r^2 \left(\dot{\theta} - 1\right)^2 dt = 0 \implies \dot{\theta} = 1 \implies \theta = t + \theta_0$ for $\theta_0 = const$.
  Simplify: Rotate axes to get $\theta_0 = 0$, then $\theta = t$.
- And $\frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2\right] dt = 0 \implies r = C \sin t$ by the equality case of Wirtinger.
- So $r = C \sin\theta$, which is polar equation of *circle* that passes through the origin. (Exercise: Obtain the Cartesian form $x^2 + \left(y - \frac{C}{2}\right)^2 = \frac{C^2}{4}$.)

# Curvature

When is a curve ...*curved*?

Definition

If $\gamma : I \to \mathbb{R}^n$ is a unit speed curve, then its curvature is $\kappa := \|\ddot{\gamma}\|$.

Interpretation: Curvature as quadratic coefficient in Taylor's theorem:

$$\gamma(t_0 + \Delta t) = \gamma(t_0) + \dot{\gamma}(t_0)\Delta t + \frac{1}{2}\ddot{\gamma}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

- Can replace $\dot{\gamma}(t_0)$ by unit tangent $\mathbf{t}(t_0) = \dot{\gamma}(t_0)$.
- $\dot{\gamma}(t_0) \cdot \dot{\gamma}(t_0) = 1 \implies 2\dot{\gamma}(t_0) \cdot \ddot{\gamma}(t_0) = 0$, so $\ddot{\gamma} \perp \dot{\gamma}$ for a unit speed curve (if $\ddot{\gamma} \neq 0$).
- Then $\ddot{\gamma} = \pm\kappa\mathbf{n}$ where $\mathbf{n}$ is unit normal vector (orthogonal to $\mathbf{t}$).
- Get $\gamma(t_0 + \Delta t) = \gamma(t_0) + \mathbf{t}(t_0)\Delta t \pm \frac{1}{2}\kappa(t_0)\mathbf{n}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3)$
- Two choices for $\mathbf{n}$: we choose it so that $\{\mathbf{t}, \mathbf{n}\}$ is *right-handed*.

# Curvature formulas: general parametrization

- Say $t$ is a general parameter for $\gamma$, and $s$ is an arclength parameter.
- Chain rule $\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} \implies \frac{d\gamma}{ds} = \frac{d\gamma/dt}{ds/dt}$.
- Chain rule again $\frac{d^2\gamma}{ds^2} = \frac{d}{ds}\left(\frac{d\gamma/dt}{ds/dt}\right) = \frac{dt}{ds}\frac{d}{dt}\left(\frac{d\gamma/dt}{ds/dt}\right) = \frac{\ddot{\gamma}(t)\dot{s}(t) - \dot{\gamma}(t)\ddot{s}(t)}{(\dot{s}(t))^3}$.
- Now use $\kappa = \left\| \frac{d^2\gamma}{ds^2} \right\|$.
- Then $\kappa = \frac{\|\ddot{\gamma}\dot{s} - \dot{\gamma}\ddot{s}\|}{|\dot{s}|^3}$.
- Then $\kappa = \frac{\left\|\ddot{\gamma}\dot{s}^2 - \dot{\gamma}\dot{s}\ddot{s}\right\|}{|\dot{s}|^4} = \frac{\|\ddot{\gamma}(\dot{\gamma}\cdot\dot{\gamma}) - \dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})\|}{\left(\|\dot{\gamma}\|^2\right)^2}$, using that $\dot{s}^2 = \left(\frac{ds}{dt}\right)^2 = \|\dot{\gamma}\|^2 = \dot{\gamma}\cdot\dot{\gamma}$ and therefore $\dot{s}\ddot{s} = \dot{\gamma}\cdot\ddot{\gamma}$.
- Finally, the "BAC-CAB rule" $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ yields $\kappa = \frac{\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\|}{\|\dot{\gamma}\|^4}$.
- Notice that $\dot{\gamma} \perp \ddot{\gamma}\times\dot{\gamma}$. Thus $\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\| = \|\dot{\gamma}\|\,\|\ddot{\gamma}\times\dot{\gamma}\|$, so $\kappa = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$.

# Example: Circle

- Circle in $\mathbb{R}^2$: $\gamma(t) = (x_0 + a\cos t, y_0 + a\sin t),\ t \in [0, 2\pi)$.
- $\dot\gamma = a(-\sin t, \cos t)$, $\ddot\gamma = -a(\cos t, \sin t)$.
- Use $\kappa = \frac{\|\ddot\gamma \times \dot\gamma\|}{\|\dot\gamma\|^3}$. Think of $\mathbb{R}^2$ as $z = 0$ plane in $\mathbb{R}^3$.
- $$\dot\gamma \times \ddot\gamma = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e}_3 \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$
  $$= \mathbf{e}_1 \begin{vmatrix} a\cos t & 0 \\ -a\sin t & 0 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} -a\sin t & 0 \\ -a\cos t & 0 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} -a\sin t & a\cos t \\ -a\cos t & -a\sin t \end{vmatrix}$$
  $$= \mathbf{e}_3 \left( a^2 \sin^2 t + a^2 \cos^2 t \right) = a^2 \mathbf{e}_3.$$
- Also, $\|\dot\gamma\| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$.
- Then $\kappa = \frac{a^2 \|\mathbf{e}_3\|}{a^3} = \frac{1}{a}$. Circles have constant curvature $= 1/\text{radius}$.

# Osculating circles

**Definition**

If a curve $\gamma : I \to \mathbb{R}^2$ has curvature $\kappa(t) \neq 0$ at point $p = \gamma(t)$, we define its *radius of curvature* at $p$ to be $\rho(t) = 1/\kappa(t)$.

The *osculating circle* to $\gamma$ at $p$
is the circle that

- passes through $p$,
- has the same tangent line as $\gamma$ at $p$,
- has radius $\rho = \frac{1}{\kappa}$, and
- lies on the concave side of $\gamma$.

# Signed curvature

- Parametrize the curve $\gamma(t)$ in $\mathbb{R}^2$.
- The direction of increasing parameter is the *orientation*.
- Define the unit tangent vector $\mathbf{t} = \dot{\gamma}/\left\|\dot{\gamma}\right\|$.
- Define the unit normal $\mathbf{n}$ by rotating $\mathbf{t}$ by $\frac{\pi}{2}$ *counter-clockwise* (also called the *right-handed sense*).
- Then the *signed curvature* $\kappa_S$ is defined by

$$\ddot{\gamma}(s) = \kappa_S \mathbf{n}$$

  where $s$ is an arclength parameter with $ds/dt > 0$ (i.e., same orientation as $t$).
- Relation to (ordinary) curvature is $\kappa := |\kappa_S|$.

# Interpretation: turning angle

**Theorem (The turning angle)**

*There is a unique smooth function $\phi$, called the turning angle, along the regular curve $\gamma$ such that $\phi(s_0) = \phi_0$ and $\mathbf{t} = (\cos\phi(s), \sin\phi(s))$.*

- Tangent vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:
  $\mathbf{t} = \dot{\gamma}(s) = (\cos\phi(s), \sin\phi(s))$
- Calculate: $\dot{\mathbf{t}} = \ddot{\gamma}(s) = \dot{\phi}(s)\,(-\sin\phi(s), \cos\phi(s))$
- Normal vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:
  $\mathbf{n} = \left(\cos\left(\phi(s) + \frac{\pi}{2}\right), \sin\left(\phi(s) + \frac{\pi}{2}\right)\right)$
  $= (-\sin\phi(s), \cos\phi(s))$
- Conclude that $\ddot{\gamma}(s) = \dot{\phi}(s)\mathbf{n}$.
- Compare to $\ddot{\gamma}(s) = \kappa_S\mathbf{n}$ to get $\boxed{\kappa_S(s) = \dot{\phi}(s).}$
- The signed curvature is the rate of change of the turning angle with respect to arclength.

# Hopf's Umlaufsatz (rotation rate)

- Integrate $\kappa_S(s) = \dot{\phi}(s)$ over curve $\gamma$.
- $\int\limits_{s_0}^{s} \kappa_S(u)du = \int\limits_{s_0}^{s} \dot{\phi}(u)du = \phi(s) - \phi(s_0)$.
- Take $\gamma$ closed, with period $T$.
- $\int\limits_{s_0}^{s_0+T} \kappa_S(u)du = \phi(s_0 + T) - \phi(s_0)$.
- But $\phi(s_0 + T) - \phi(s_0) = 2\pi k$, $k \in \mathbb{Z}$.
- In fact, can argue that $k = \pm 1$ if curve traversed once; $k$ is the *winding number*.

Theorem (Hopf's Umlaufsatz)

*The total curvature of a closed curve of period $T$ is* $\int\limits_{s_0}^{s_0+T} \kappa_S(u)du = \pm 2\pi$.

# A useful formula

- $\mathbf{t} = \frac{\partial \gamma}{\partial s}$ = unit tangent vector.
- Last slide: $\frac{\partial^2 \gamma}{\partial s^2} = \kappa_S \mathbf{n}$.
- Then $\frac{\partial^3 \gamma}{\partial s^3} = \kappa_S \frac{\partial \mathbf{n}}{\partial s} + \frac{\partial \kappa_S}{\partial s} \mathbf{n}$.
- And then $\frac{\partial \gamma}{\partial s} \cdot \frac{\partial^3 \gamma}{\partial s^3} = \kappa_S \mathbf{t} \cdot \frac{\partial \mathbf{n}}{\partial s} + \frac{\partial \kappa_S}{\partial s} \mathbf{t} \cdot \mathbf{n} = \kappa_S \mathbf{t} \cdot \frac{\partial \mathbf{n}}{\partial s} = \kappa_S \frac{\partial \gamma}{\partial s} \cdot \frac{\partial \mathbf{n}}{\partial s}$.
- But $\frac{\partial \gamma}{\partial s} \cdot \frac{\partial^3 \gamma}{\partial s^3} = \frac{\partial}{\partial s} \left( \frac{\partial \gamma}{\partial s} \cdot \frac{\partial^2 \gamma}{\partial s^2} \right) - \frac{\partial^2 \gamma}{\partial s^2} \cdot \frac{\partial^2 \gamma}{\partial s^2} = -\frac{\partial^2 \gamma}{\partial s^2} \cdot \frac{\partial^2 \gamma}{\partial s^2} = -\kappa_S^2$.
- Comparing the last two lines, then

$$\frac{\partial \gamma}{\partial s} \cdot \frac{\partial \mathbf{n}}{\partial s} = -\kappa_S .$$

# Application: Curve shortening flow (CSF)

- From here on, I will write $\kappa$ when I mean $\kappa_S$.
- Let $\gamma(u)$ be a parametrized curve. Let $\mathbf{N}(u)$ be the normal vector to the curve. Let $\gamma(t,s)$ be a smooth family of such curves, one for each value of $t \in (-T, T)$ for some $T \in (0, \infty]$. The curve shortening flow is

$$\frac{\partial \gamma}{\partial t} = \kappa \mathbf{N}(t).$$

- A more useful definition uses the arclength parametrization and the inner product $\langle \cdot, \cdot \rangle$ (dot product on $\mathbb{R}^2$):

$$\left\langle \mathbf{N}, \frac{\partial \gamma}{\partial t} \right\rangle = \kappa.$$

- Online demonstration at https://a.carapetis.com/csf/

# A simple curve shortening flow: Circle

- $\gamma(s) = a\left(\cos\frac{s}{a}, \sin\frac{s}{a}\right)$ (we'll let $a = a(t)$).
- $\gamma'(s) = \mathbf{T} = \left(-\sin\frac{s}{a}, \cos\frac{s}{a}\right)$.
- $\gamma''(s) = \kappa\mathbf{N} = -\frac{1}{a}\left(\cos\frac{s}{a}, \sin\frac{s}{a}\right)$.
- $\dot{\gamma} = \frac{\partial\gamma}{\partial t} = \dot{a}\left(\cos\frac{s}{a}, \sin\frac{s}{a}\right) - \frac{s\dot{a}}{a}\left(-\sin\frac{s}{a}, \cos\frac{s}{a}\right)$.
- $\langle\mathbf{N}, \dot{\gamma}\rangle = -\dot{a}$.
- But by CSF, $\langle\mathbf{N}, \dot{\gamma}\rangle = \kappa = \frac{1}{a}$.
- Therefore $-\dot{a} = \frac{1}{a}$, so $2a\dot{a} \equiv \frac{\partial}{\partial t}\left(a^2\right) = -2$.
- If the initial radius is $a(0) = a_0$, then we get $a^2(t) = a_0^2 - 2t$.
- The flow is $\gamma(s,t) = \sqrt{a_0^2 - 2t}\left(\cos\frac{s}{\sqrt{a_0^2-2t}}, \sin\frac{s}{\sqrt{a_0^2-2t}}\right)$
- The circle disappears at time $t = \frac{a_0^2}{2}$.

# A CSF soliton: the "grim reaper"

- $\gamma(u,t) = (u, t - \log\cos u)$.
- $\gamma' = (1, \tan u)$, so $\frac{ds}{du} = \|\gamma'\| = \sec u$.
- $\mathbf{T} = \frac{(1,\tan u)}{\sqrt{1+\tan^2 u}} = (\cos u, \sin u)$.
- $\mathbf{N} = (-\sin u, \cos u)$.
- $\dot\gamma = (0,1)$ (velocity is vertical and constant).
- $\langle \mathbf{N}, \dot\gamma \rangle = \cos u$.
- So this is a CSF if $\kappa = \cos u$.
- By direct computation, we have $\kappa\mathbf{N} = \frac{\partial}{\partial s}\mathbf{T} = \frac{\partial u}{\partial s}\frac{\partial \mathbf{T}}{\partial u} = \cos u(-\sin u, \cos u)$ so indeed $\kappa = \cos u$!

# How to prove the isoperimetric inequality with CSF

Say that for each $t \in [0, T)$, $\gamma(u, t)$ is a closed curve with arclength function $s = s(u, t)$ and say that $\frac{\partial \gamma}{\partial t} = \kappa \mathbf{N}$.

Four claims:

1. The length $L(t)$ of the curve obeys $\frac{dL}{dt} = -\int \kappa^2 ds$.
2. The enclosed area $A(t)$ obeys $\frac{dA}{dt} = -\int \kappa ds$.
   - Then $\frac{dA}{dt} = -2\pi$ by Hopf's Umlaufsatz.
3. Any such flow ends in finite time, say $T$.
4. As $t \nearrow T$, the curvature approaches a uniform function of time: $\kappa(s, t) \nearrow k(t) \to \infty$.
   - Then the curve becomes a circle of radius $\frac{1}{k(t)}$.

# Sketch of proof

If the Claims are true, then

$$
\begin{aligned}
\frac{d}{dt}\left(L^2 - 4\pi A\right) &= 2L\frac{dL}{dt} - 4\pi\frac{dA}{dt} \\
&= -2L\int_\gamma \kappa^2 ds + 4\pi\int_\gamma \kappa ds \\
&\le -2\left(\int_\gamma \kappa ds\right)^2 + 4\pi\int_\gamma \kappa ds \text{ (Hölder inequality)} \\
&= -2\left(\int_\gamma \kappa ds\right)^2 + 2\left(\int_\gamma \kappa ds\right)^2 \text{ (Umlaufsatz)} \\
&= 0.
\end{aligned}
$$

and since $L^2(t) - 4\pi A(t) \to 0$ as $t \nearrow T$ by Claim 3, then at every $0 \le t < T$ we have $L^2 - 4\pi A \ge 0$. So $L^2 - 4\pi A \ge 0$ for the initial curve $\gamma(u, 0)$.

- This is the isoperimetric inequality. Now we must prove the Claims, which are facts about CSF.

# Claim 1: $\frac{dL}{dt} = -\int \kappa^2 ds$

- If $u$ is a parameter for the curve $\gamma(u) = (x(u), y(u))$ and $s$ is an arclength parameter then $ds = \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2} du = \|\gamma'(u)\| \, du$. Then

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\left\|\frac{\partial\gamma}{\partial u}\right\|^2\right) =& 2\frac{\partial\gamma}{\partial u}\cdot\frac{\partial}{\partial u}\frac{\partial\gamma}{\partial t} \\
=& 2\frac{\partial\gamma}{\partial u}\cdot\frac{\partial}{\partial u}(\kappa\mathbf{n}) \text{ by the CSF equation} \\
=& 2\left\|\frac{\partial\gamma}{\partial u}\right\|^2\frac{\partial\gamma}{\partial s}\cdot\frac{\partial}{\partial s}(\kappa\mathbf{n}) \text{ since } \frac{\partial}{\partial u} = \left\|\frac{\partial\gamma}{\partial u}\right\|\frac{\partial}{\partial s} \\
=& -2\kappa^2\left\|\frac{\partial\gamma}{\partial u}\right\|^2,
\end{aligned}$$

using that $\frac{\partial\gamma}{\partial s}\cdot\frac{\partial}{\partial s}(\kappa\mathbf{n}) = \mathbf{t}\cdot\left(\frac{\partial\kappa}{\partial s}\mathbf{n} + \kappa\frac{\partial\mathbf{n}}{\partial s}\right) = -\kappa^2$ since $\mathbf{t}\cdot\mathbf{n} = 0$ and $\mathbf{t}\cdot\frac{\partial\mathbf{n}}{\partial s} = -\kappa$ by our earlier "useful equation".

# Claim 1 continued: $\frac{dL}{dt} = -\int \kappa^2 ds$

Then

$$\frac{\partial}{\partial t}\left(\left\|\frac{\partial\gamma}{\partial u}\right\|\right) = -\kappa^2\left\|\frac{\partial\gamma}{\partial u}\right\|,$$

and so

$$\begin{aligned}
\frac{dL}{dt} &= \frac{d}{dt}\int ds \\
&= \frac{d}{dt}\int\left\|\frac{\partial\gamma}{\partial u}\right\| du \\
&= \int \frac{\partial}{\partial t}\left\|\frac{\partial\gamma}{\partial u}\right\| du \\
&= -\int \kappa^2\left\|\frac{\partial\gamma}{\partial u}\right\| du \\
&= -\int \kappa^2 ds.
\end{aligned}$$

QED

# Claim 2: $\frac{dA}{dt} = -\int \kappa ds$

- Begin with $A = \frac{1}{2}\int_R (xdy - ydx) = \frac{1}{2}\int_\gamma (x, y) \cdot \left(\frac{\partial y}{\partial s}, -\frac{\partial x}{\partial s}\right) ds = \frac{1}{2}\int_\gamma \gamma \cdot \mathbf{n} ds$.
- Then $\frac{dA}{dt} = \frac{1}{2}\int_\gamma \left(\frac{\partial \gamma}{\partial t} \cdot \mathbf{n} + \gamma \cdot \frac{\partial \mathbf{n}}{\partial t}\right) ds - \frac{1}{2}\int \gamma \cdot \mathbf{n}\kappa^2 ds$, where the $-\kappa^2$ arises in the same way it did when proving Claim 1.
- Now $\frac{\partial \gamma}{\partial t} \cdot \mathbf{n} = -\kappa$ from the CSF equation.
- Can show that $\frac{\partial \mathbf{n}}{\partial t} = \frac{\partial \kappa}{\partial s}\mathbf{t}$ by differentiating the flow equation.
- Then

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}\int_\gamma \left(-\kappa + \frac{\partial \kappa}{\partial s}\gamma \cdot \mathbf{t} - \gamma \cdot \mathbf{n}\kappa^2\right) ds \\ &= \frac{1}{2}\int_\gamma \left(-\kappa - \kappa\frac{\partial \gamma}{\partial s} \cdot \mathbf{t} - \kappa\gamma \cdot \frac{\partial \mathbf{t}}{\partial s} - \gamma \cdot \mathbf{n}\kappa^2\right) ds \\ &= \frac{1}{2}\int_\gamma \left(-2\kappa + \kappa^2\gamma \cdot \mathbf{n} - \gamma \cdot \mathbf{n}\kappa^2\right) ds \text{ using } \frac{\partial^2 \gamma}{\partial s^2} = \frac{\partial \mathbf{t}}{\partial s} = -\kappa\mathbf{n} \\ &= -\int_\gamma \kappa ds \qquad \text{QED} \end{aligned} \tag{4}$$

# Conclusion

- I will omit the proof of Claim 3. See
  - ME Gage, *An isoperimetric inequality with applications to curve-shortening*, Duke Math J 50 (1983) 1225–1229.
  - MA Grayson, *The heat equation shrinks embedded plane curves to round points*, Journal of Differential Geometry 26(2) (1987) 285–314.
  - B Andrews, B Chow, C Guenther, and M Langford, *Extrinsic geometric flows*, Graduate Studies in Mathematics 206 (American Mathematical Society, Providence, 2020).
- There are many other problems that can be treated using curvature flows.
  - One example: Can the isoperimetric inequality on the sphere be proved this way?