Lower Ricci curvature bounds via optimal transport

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Plan of the lectures

Today: introductory material.

- What is optimal transport?
- What is known? What sort of mathematics is involved?
- Why should I care? What can I do with it? Applications?

Friday: a deeper look at one selected topic. At the end of today's talk, we can vote to decide on the topic. Choices include:

- Matching theory (economics): what sort of patterns emerge when agents match together (for instance, workers and firms on the labour market, or husbands and wives on the marriage market).
- Density functional theory (physics/chemistry): how does a system of electrons organize itself to minimize interaction energy.
- Curvature and entropy (geometry): How does curvature relate to the behavior of densities along interpolations?

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Analytic vs synthetic

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Properties:

1. local, easy to check
2. general (does not *require* smoothness), stable (under pointwise convergence), *implies* some smoothness Alexandrov's theorem:
   - 2 $\implies \frac{d^2 f}{dx^2}$ exists a.e.

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(if u and v are orthogonal). Non-negative sectional curvature $K(u, v) \geq 0 \iff$ geodesics stay closer together than in Euclidean space.

Synthetic formulation: Alexandrov spaces

Theorem (Alexandrov)

M has non-negative sectional curvature iff, for all small geodesic triangles p, q, r, and any $x \in [r, q]$, we have

$$d(x, p) \geq d(X, P) = |X - P|$$

- P,Q,R is the Euclidean comparision triangle.
- extends notion of non-negative sectional curvature to geodesic spaces. Geodesics spaces with non-negative sectional curvature are called Alexandrov spaces.
- Gromov-Hausdorff stable
- Theory of Alexandrov spaces developed by Burago, Perelman, Petrunin, Ohta (and Alexandrov):
  - *almost smooth* structure (like a manifold with convex-concave transition maps); singluar points have measure zero.

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$Ric \geq 0$ (ie $Ric(v, v) \geq 0$ for all v) means that objects look bigger than they are (light test).

Motivation

$Ric \geq 0$ implies:

1. Bonnet-Myers theorem (bounds on diameter) (need strict inequality)
2. Bishop-Gromov inequality (volume distortion of geodesic balls).
3. Isoperimetric inequalities
4. Brunn-Minkowski inequalities
5. Li-Yau estimates on heat kernels

Theorem (Gromov)

The set of compact manifolds with diameter $\leq D$ with $Ric \geq 0$ is precompact (Gromov-Hausdorff distance)

Limit points may not be smooth....do they satisfy $Ric \geq 0$ (in some sense)?

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If F is smooth and injective, we have the change of variables formula:

$$|det(DF(x))| = f(x)/g(F(x))$$

Existence and uniqueness of a solution

For $c(x,y) = |x-y|^2$ (on $\mathbb{R}^n$) or $d^2(x,y)$ (on M):

Theorem (Brenier/McCann)

There exists a unique solution F to Monge's problem.

- When $M \subseteq \mathbb{R}^n$, $F(x) = \nabla u(x)$, u convex.
- More generally, $F(x) = exp_x(\nabla u(x))$, where u is d^2-convex.

Note: $\exp_x(v)$ starts at x and then moves distance $|v|$ along the tangent vector v. In $\mathbb{R}^n$, $\exp_x(v) = x + v$

Optimal transport formulation of Ricci curvature ≥ 0

Optimal transport gives a nonlinear interpolation between measures:

$$f_t(x)dx := exp_x(t\nabla u)_\#(f(x)dx)$$

The entropy functional, along these interpolations detects Ricci curvature:

$$H(f) := \int \ln(f(x))f(x)dx$$

The physical entropy, $-H$, measures how spread out, or random, the density is.
The lazy gas experiment:

Theorem (Otto-Villani/Cordero-Erausquin-McCann-Schmuckenschlager/Sturm-von Renesse)

$Ric \geq 0$ iff $t \mapsto H(f_t)$ is convex for any f, g.

Optimal transport makes sense on very general spaces, and this definition extends to metric measured length spaces.

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$(\implies)$

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- Apply convexity under the integral sign. The composition of a concave and convex-decreasing function is convex.

Consequences of optimal transport formulation

- Stable under measured Gromov-Hausdorff convergence (Lott-Villani/Sturm)
- Brunn-Minkowski, isoperimetric inequalities, Poincare inequalities (Lott-Villani/Sturm)
- Bishop-Gromov, Bonnet-Myers type comparison theorems (Lott-Villani/Sturm)
- Compatibility with Alexandrov space: weak sectional curvature bounds imply weak Ricci curvature bounds (an Alexandrov space is a Lott-Villani/Sturm space) (Petrunin)

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