

$X \subseteq \mathbb{C}^n$ algebraic set

$$\mathcal{O}(X) = \{ f: X \rightarrow \mathbb{C} \text{ st. } f \text{ regular} \}$$

Def: $\varphi: X \rightarrow Y$ is called a morphism if

$$\forall f \in \mathcal{O}(Y), \quad f \circ \varphi \in \mathcal{O}(X)$$

$$f \circ \varphi(x) = f(\varphi(x))$$

$$\begin{array}{c} X \xrightarrow{\varphi} Y \xrightarrow{f} \mathbb{C} \\ \quad \quad \quad \searrow \quad \quad \quad \nearrow \\ \quad \quad \quad f \circ \varphi \end{array}$$

Note: $f, g \in \mathcal{O}(Y) \quad (f+g) \circ \varphi = f \circ \varphi + g \circ \varphi$

$$(fg) \circ \varphi = (f \circ \varphi)(g \circ \varphi)$$

$f \text{ const} \Rightarrow f \circ \varphi \text{ is the same const.}$

$\rightsquigarrow$ φ morphism $\Rightarrow \varphi^*: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is a map of $\mathbb{C}$ -algebras
 $f \mapsto f \circ \varphi$

φ^* called co-morphism of φ .

Fact: if $\alpha: \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is a map of $\mathbb{C}$ -algebras
then $\exists! \varphi: X \rightarrow Y$ morphism s.t. $\alpha = \varphi^*$.

$\varphi: X \rightarrow Y$ $p \in X$ Fact: $m_p = \{ f \in \mathcal{O}(X) \mid f(p) = 0 \}$

$m_p \subseteq \mathcal{O}(X)$ max ideal in $\mathcal{O}(X)$

$\alpha^{-1}(m_p) \subseteq \mathcal{O}(Y)$ is a max. ideal of $\mathcal{O}(Y)$

$\Rightarrow \exists! Q = Z(\alpha^{-1}(m_p)) = Q = \{q\}$,

$\varphi(p) = q$.

Affine var. form a category. $\circ \text{Aff}_{\mathbb{C}} \rightarrow \mathbb{C}\text{-alg.}$

$$X \mapsto \mathcal{O}(X)$$

- $\varphi: X \rightarrow Y$ is an isomorphism if
 - φ is a bijective morphism
 - and • φ^{-1} is a morphism.

Equivalently: φ^* is an isomorphism.

- $A_1^n := \mathbb{C}^n$ when viewed as an affine variety.

$$\bullet \quad f: X \rightarrow A_1^1 = \mathbb{C} \iff f \in \mathcal{O}(X)$$

$$\text{Mor}(X, A_1^1) \iff \text{Hom}(\mathcal{O}(A_1^1), \mathcal{O}(X)) \quad \mathbb{C}[x] \rightarrow \mathcal{O}(X)$$

$\mathcal{O}(X)$ $(\varphi: \mathbb{C}[X] \rightarrow \mathcal{O}(X))$  $\varphi(x)$

$f: X \rightarrow A^1$ morphism

$$\varphi^*(x) = x \circ f = f$$

$\varphi: X \rightarrow A^1^n \hookrightarrow \mathcal{O}(X)^n$

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \quad \varphi_i = \varphi^*(x_i)$$

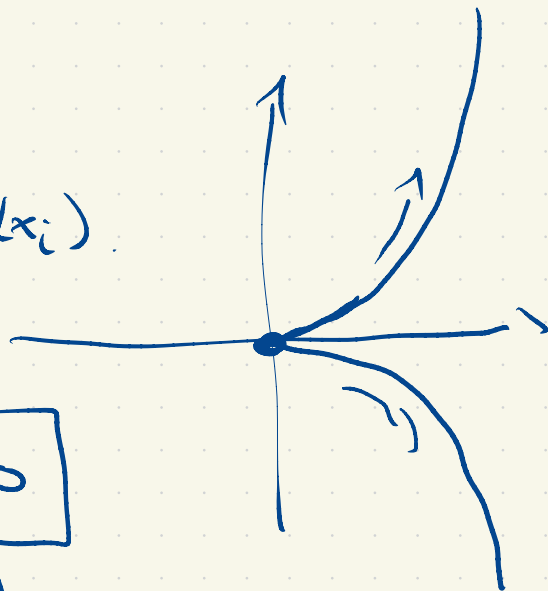
• $\varphi \quad C = \mathbb{Z}(x^3 - y^2) \in A^2$

$$\varphi: A^1 \rightarrow C$$

$$t \mapsto (t^2, t^3)$$

$$\boxed{x^3 - y^2 = 0}$$

$$(t, \pm t^{3/2})$$



Fact: φ bijection.

$\Leftarrow (x, y) \in C \quad (x, y) \neq (0, 0)$
then $x \neq 0$

$$(x, y) = \varphi\left(\frac{y}{x}\right),$$

φ is not an isomorphism.

$$\varphi^*(\mathcal{O}(C)) \cong \mathbb{C}[t] = \mathcal{O}(A')$$

gen. by $\varphi^*(x), \varphi^*(y)$
 $\parallel$ $\parallel$
 t^2 t^3

$$t \notin \varphi^*(\mathcal{O}(C))$$

$\Rightarrow \varphi^*$ not an isom.

$$\mathcal{O}(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{I(X)}$$

$$\text{Mor}(Y, X) \stackrel{!}{=} \text{Hom}(\mathcal{O}_X, \mathcal{O}_Y)$$

R $\mathbb{C}$ -alg. $\text{Hom}(\mathcal{O}(X), R)$

$$\begin{array}{ccc} & \mathbb{C}[x_1, \dots, x_n] / \mathcal{I}(X) & \xrightarrow{\quad} R \\ & \uparrow & \nearrow \\ & \mathbb{C}[x_1, \dots, x_n] & \end{array}$$

$$\text{Hom}(\mathcal{O}(X), R) = \{ \varphi : \mathbb{C}[x_1, \dots, x_n] \rightarrow R \mid \varphi(\mathcal{I}(X)) = 0 \}$$

$$\varphi(f(x_1, \dots, x_n)) = \underline{\underline{f(\varphi(x_1), \varphi(x_2), \dots, \varphi(x_n))}}$$

$$\text{Hom}(\mathbb{C}[x_1, \dots, x_n], R) \stackrel{!}{=} R^n$$

$(r_1, \dots, r_n)$ corresponds to $\mathcal{O}(X) \rightarrow R$

$$\Leftrightarrow f(r_1, \dots, r_n) = 0 \quad \forall f \in \mathcal{I}(X).$$

$$\text{Hom}(\mathcal{O}(X), \mathbb{R}) \xrightarrow{!} \{x \in \mathbb{R}^n \mid f(x) = 0 \quad \forall f \in \mathcal{I}(X)\}$$

$$\overset{!!}{X(\mathbb{R})} \quad \text{"}\mathbb{R}\text{-valued pts of } X\text{"}$$

$$X(\mathbb{C}) = X.$$

$$x^n + y^n - z^n$$

- $X \subseteq \mathbb{C}^n$ affine var.

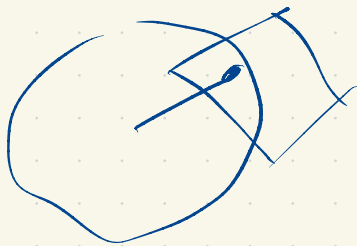
$$p \in X: \quad T_p(X) = \text{tangent space}$$

$$:= \{v \in \mathbb{C}^n \mid df_p(v) = 0 \quad \forall f \in \mathcal{I}(X)\}$$

$$\sum \frac{\partial f}{\partial x_i}(p) \cdot v_i$$

- $T_p(\mathbb{A}^n) = \mathbb{C}^n$

- $\emptyset \quad S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \quad f = x^2 + y^2 + z^2 - 1$
 $T_p(S^2): \quad df_p = 2x dx + 2y dy + 2z dz$ generates $I(S^2)$
 $df_p(v) = 0 \Leftrightarrow p \cdot v = 0.$



- $X = Z(xy) \subset \mathbb{A}^2 \quad f = xy \quad df = y dx + x dy.$



$$T_p(X) = \begin{cases} p \in x\text{-axis} \setminus y\text{-axis} \xrightarrow{y=0, x \neq 0} x\text{-axis} \\ p \in y\text{-axis} \setminus x\text{-axis} \xrightarrow{x=0, y \neq 0} y\text{-axis} \\ p = (0,0) \xrightarrow{} \mathbb{C}^2 \end{cases}$$

- $C = \mathbb{Z}(x^3 - y^2)$ $f = x^3 - y^2$ $df = 3x^2 dx - 2y dy$
- X a var. $\dim_p X = \text{max length of a chain}$
 $\{p\} \subsetneq X_1 \subsetneq X_2 \subsetneq \dots \subsetneq X_m$ of closed irred. subsets.
 $\text{length} := m$

$$\dim_p \mathbb{A}^n = n \quad \forall p \in \mathbb{A}^n.$$

X irreducible: $\dim_p(X)$ independent of p .
 $\quad \quad \quad =: \dim(X).$

X not irred. $X = \bigcup_i X_i$ $\dim_p(X) = \max_{p \in X_i} \dim X_i$

$$\dim(X) := \max \dim X_i$$

Fact: • $\dim T_p(X) \geq \dim_p(X)$

• equality on an open dense subset of X .

Def: X called smooth at p or nonsingular at p if
 $\dim T_p(X) = \dim_p(X).$

otherwise X is called singular at p .

• Alg groups

Def: An (affine) alg. group G is an affine variety G together with a morphism $m: G \times G \rightarrow G$

s.t. m is the mult. of a group structure on G

and $\iota: G \rightarrow G$ is a morphism.
 $g \mapsto g^{-1}$

• Examples:

$$GL_n(\mathbb{C}) = \{ A \in M_n(\mathbb{C}) \mid \det(A) \neq 0 \}$$

$$M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$$

Zariski-open subset of $M_n(\mathbb{C})$.

We identify $GL_n(\mathbb{C})$ with

$$\hat{G} \equiv \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \lambda \end{array} \right) \in M_{n+1}(\mathbb{C}) \mid \det(g) \cdot \lambda = 1 \right\}$$

$$\begin{array}{ccc} GL_n(\mathbb{C}) & \xrightarrow{1:1} & \hat{G} \\ g & \mapsto & \left(\begin{array}{c|c} g & \\ \hline & \det(g)^{-1} \end{array} \right) \end{array}$$

$$\mathcal{O}(GL_n(\mathbb{C})) = \mathbb{C}[x_{ij}, y_{ij} \mid 1 \leq i, j \leq n] / (\det \cdot y - 1)$$

$$= \mathbb{C}[x_{ij}, \frac{1}{\det}]$$

$$f: GL_n(\mathbb{C}) \rightarrow \mathbb{C} \text{ regular}$$

$$\Leftrightarrow f = \frac{P(x)}{\det()^m}$$

$$m: \underbrace{GL_n \times GL_n} \rightarrow GL_n$$

$$\mathbb{Q}[x_{ij}, \det^{-1}, y_{ij}, \tilde{\det}^{-1}]$$

mult of two matrices is polynomial in the entries

$m^*(x_{ij})$ is regular.

$$m^*(x_{ij}) = \sum_{k=1}^n \tilde{\tau} x_{ik} g_{kj}$$

regular

m morphism.

$\tilde{\tau}$ is regular:

$$\tilde{\tau}(g) = g^{-1}$$

$$g^{-1} = \frac{1}{\det g} \text{Adj}(g)$$

$$g \cdot \text{Adj}(g) = \begin{bmatrix} \det g & & \\ & \det(g) & \\ & & \ddots \\ & & & \det(g) \end{bmatrix}$$

entries of g^{-1} are regular in the entries of g .

$\Rightarrow \tilde{\tau}$ is a morphism.

"special linear group"

$$\bullet \rightarrow SL_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid \det g = 1 \}$$

$$\bullet \quad O_n(\mathbb{C}) = \{ g \in GL_n(\mathbb{C}) \mid g^T g = \mathbb{I} \}$$

orthogonal group

$$\rightarrow SO_n(\mathbb{C}) = \{ g \in O_n(\mathbb{C}) \mid \det g = 1 \}$$

- symplectic group: $Sp_{2n}(\mathbb{C}) = \{g \in GL_{2n}(\mathbb{C}) \mid g^T J g = J\}$

$$J = \left(\begin{array}{c|c} & I_n \\ \hline -I_n & \end{array} \right)$$

- $T_n = \{g \in GL_n(\mathbb{C}) \mid g \text{ diagonal}\}$

- $B_n = \{g \in GL_n(\mathbb{C}) \mid g \text{ upper triangular}\}$

- $U_n = \{g \in GL_n(\mathbb{C}) \mid g \text{ upper triangular, all diag. entries} = 1\}$

- a few exceptional groups: G_2, F_4, E_6, E_7, E_8 .

Fact: · alg. groups are always smooth everywhere.

$$\lambda_g: G \rightarrow G$$
$$\lambda_g(x) = gx$$

inverse $\lambda_g^{-1} = \lambda_{g^{-1}}$

isomorphism

$$\varphi: X \rightarrow Y$$

$$p \in X: \mathcal{D}_p \varphi: T_p(X) \rightarrow T_{\varphi(p)}(Y).$$

$$\varphi \text{ isom.} \Rightarrow \mathcal{D}_p \varphi \text{ is an isomorphism}$$

$$\mathcal{D}_e \lambda_g: T_e(G) \xrightarrow{\sim} T_g(G)$$

$$\Rightarrow \dim T_g(G) = \dim G \quad \forall g.$$

$$T_e(G): \quad \text{Int}(g)(x) = g x g^{-1}$$

$$\text{Ad}(g) := D_e \text{Int}(g) : T_e(G) \rightarrow T_e(G)$$

$$[X, Y] := \frac{\partial \text{Ad}}{\partial X}(Y)$$

$$G = GL_n: \quad T_e(GL_n) = \underline{M_n(\mathbb{C})}$$

$$\hat{G} = \left\{ \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \lambda \end{array} \right) \mid \det(g) \cdot \lambda = 1 \right\}$$

$$T_e(\hat{G}) = \left\{ \left(\begin{array}{c|c} A & \\ \hline & t \end{array} \right) \mid \text{trace } A + t = 0 \right\}$$

$$\text{trace}(t) = a_{11} + a_{22} + \dots + a_{nn}$$

$$[X, Y] = XY - YX$$

$$G \subset GL_n(\mathbb{C})$$

$T_e(G) \subset M_n(\mathbb{C})$ is stable under $[\cdot, \cdot]$.

The Lie algebra of G .

$$\text{Lie}(G) := T_e(G) \text{ with } [\cdot, \cdot].$$

G ^{irreducible} commutative (i.e. $gh = hg \forall g, h \in G$) $\Leftrightarrow [\cdot, \cdot] = 0$.

classify simple complex Lie algebras

- A_n

$$G_2, F_4, E_6, E_7, E_8$$

- B_n

- C_n

- D_n

X affine variety

G alg group

" G acts on X " $\leftrightarrow \mu: G \times X \rightarrow X$ morphism.

$$gx := \mu(g, x)$$

$$\bullet e x = x \quad \forall x \in X$$

$$\bullet g(hx) = (gh)x \quad \forall x \in X, g, h \in G$$

$$\left(\bullet \mu(e, \cdot) = \text{id}_X \right.$$

$$\left. \bullet \mu(g, \mu(h, \cdot)) = \mu(gh, \cdot) \right)$$

• G -orbit: subset of X of the form $\{gx \mid g \in G\}$

• Stabilizer of x : $G_x = \{g \in G \mid gx = x\}$ closed subgroup.

Example: • GL_n acts on A^n

$$g \cdot x = gx$$

• GL_n acts also on $M_n(\mathbb{C})$

$$G \quad \mu(g, A) = gAg^{-1}$$

A similar to B

$$\Leftrightarrow \exists g : gAg^{-1} = B.$$

G acts on X then G acts on $\mathcal{O}(X)$.

$$gf(x) = f(g^{-1}x)$$

Put $A := \mathcal{O}(X)^G$ sub alg. of $\mathcal{O}(X)$.

If A is finitely generated:

$$\mathbb{C}[x_1, \dots, x_n]/I$$

then A is the ring of functions of an affine var.
 $(Z(I) \subseteq \mathbb{C}^n)$ Y .

$$\pi: X \rightarrow Y \subseteq \mathbb{C}^n \quad \pi^*: A \hookrightarrow \mathcal{O}(X).$$

Note: π constant on any G -orbit.

π as a map to $\mathbb{C}^n$ is given by n elements of $A = \mathcal{O}(X)^G$

Y is called the quotient of X by G denoted $\boxed{X//G}$.

Note $X/G = \text{set of } G\text{-orbits}$

$GL_n(\mathbb{C}) \curvearrowright \mathbb{C}$ orbits: $\begin{cases} \bullet \text{ all nonzero complex numbers} \\ \bullet 0 \end{cases}$

~~ex~~ $X//G$ parameterizes the closed orbits for reasonable groups

$x \in X//G$ $\pi^{-1}(x)$ G -stable closed subset of X .
contains a unique closed orbit.

$$GL_n \curvearrowright M_n(\mathbb{C})$$

Are there invariants

$f_1, \dots, f_m$ s.t.

$$f_i(A) = f_i(B) \quad \forall i \quad \text{then } A \text{ is sim to } B.$$

• $f: M_n(\mathbb{C}) \rightarrow \mathbb{C}$ invariant under GL_n

$\Leftrightarrow$ f is a polynomial in the coeff. of the characteristic equation

$$\begin{aligned}
 P_A(x) &= \det(xI - A) = \underline{x^n \pm \text{trace}(A)x^{n-1} \pm \dots \pm \det(A)} \\
 &= x^n - \sigma_1 x^{n-1} \pm \dots + \sigma_n
 \end{aligned}$$

$$\sigma_i \in \mathcal{O}(M_n)^{GL_n}$$

Every invariant is a polynomial in $\sigma_1, \dots, \sigma_n$.

$$M_n \rightarrow \mathbb{C}^n$$

$$A \mapsto (\sigma_1(A), \sigma_2(A), \dots, \sigma_n(A))$$

$$\uparrow \quad \nearrow \quad \nearrow$$

elementary symmetric poly. in the eigenvalues of A .

if A has distinct eigenvalues: $\pi^{-1}(\pi(A)) = GL_n \cdot A$

if not then there is a unique closed orbit

and orbits of matrices such as

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \\ 0 & & & & \ddots \end{pmatrix}$$