Algebraic Geometry

Algebra:
- Linear Algebra
- groups
- rings / fields
- equations

Commutative Algebra $\Longleftrightarrow$ Algebraic Geometry

Linear Algebra: solutions of systems of linear equations.

Algebraic Geometry: " " " " "

$\underline{\underline{K}}$ field

$$\begin{cases} \mathbb{R}, \mathbb{Q}, \mathbb{C} \\ \mathbb{F}_2, \mathbb{F}_3, \ldots \end{cases}$$

$1 + x^2$

$$\boxed{x^n + y^n + z^n = 0}$$

$K[x_1, \ldots, x_n]$ polynomial ring in n-variables

$\cup$

$$f = \sum_{i_1, i_2, \ldots, i_n} c_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$ finite sum.

$\underline{K = \mathbb{F}_2} := \{0, 1\}$ $\quad x = x^2 \;\; \forall x \in \mathbb{F}_2$.

K infinite. May think of $K[x_1, \ldots, x_n]$ as K-valued functions on K^n.

$$f(a_1,\dots,a_n) = \sum c_{i_1 i_2 \cdots i_n} a_1^{i_1} \cdots a_n^{i_n}$$

<u>$K = \mathbb{C}$:</u> Objects of interest: Solutions of equations

$$f(x) = 0 \quad (f \in S \subseteq \mathbb{C}[x_1,\dots,x_n])$$

$1 + x^2 = 0$ has no solutions in $\mathbb{R}$

<u>Def:</u> A subset $A \subseteq \mathbb{C}^n$ is called <u>algebraic</u> if

$$A = Z(S) = \{ x \in \mathbb{C}^n \mid f(x) = 0 \ \forall f \in S \}$$

for some $S \subseteq \mathbb{C}[x_1,\dots,x_n]$.

<u>Ex:</u>

- $\mathbb{C}^n = Z(0)$.
- $\emptyset = Z(1)$

- $S^1 = \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2-1=0\}$

- $S^n = \{x \in \mathbb{C}^n \mid x_1^2+x_2^2+\dots+x_n^2-1=0\}$

- $S \subseteq \mathbb{C}[x_1,\dots,x_n]$ $\qquad Z(S) = ?$

<u>Def</u>: $I \subseteq \mathbb{C}[x_1,\dots,x_n]$ is called an <u>ideal</u> if

·) $0 \in I$.

··) I closed under addition: $f,g \in I \Rightarrow f+g \in I$.

···) $f \in I$, $g \in \mathbb{C}[x_1,\dots,x_n] \Rightarrow gf \in I$

<u>Ex</u>: · $f \in \mathbb{C}[x_1,\dots,x_n]$: $(f) := \{gf \mid g \in \mathbb{C}[x_1,\dots,x_n]\}$.

"principal ideals"

- $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$: $(f_1, \dots, f_m) := \{\sum r_i f_i \mid r_i \in \mathbb{C}[x_1, \dots, x_n]\}$

$X \subseteq \mathbb{C}^n$ $\quad I(X) := \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f(x_1, \dots, x_n) = 0 \ \forall (x_1, \dots, x_n) \in X \}$

<u>Fact</u> $I(X)$ is an ideal:
- $0 \in I(X)$
- $f, g \in I(X) \Rightarrow f + g = 0$ on X.
- $f \in I(X)$, $g \in \mathbb{C}[x_1, \dots, x_n]$: $gf \in I(X)$.

$S \subseteq \mathbb{C}[x_1, \dots, x_n]$ $\quad \langle S \rangle :=$ ideal generated by S

$= \{ \sum r_i s_i \mid s_i \in S, r_i \in \mathbb{C}[x_1, \dots, x_n] \}$

finite

$$Z(S) = Z(\langle S \rangle)$$

Ex: alg. sets in $\mathbb{C}$.

$f \in \mathbb{C}[x]$: $\quad Z(f) = \begin{cases} \cdot \text{ finitely many } (\leq \deg f) \\ \cdot\ \emptyset \qquad (f = \text{const} \neq 0) \\ \cdot\ \mathbb{C} \qquad f = 0. \end{cases}$

$A = \{a_1, \dots, a_m \in \mathbb{C}\} \qquad A = Z(f)$

$$f = (x - a_1)(x - a_2) \cdots (x - a_m).$$

Fact: Every ideal in $\mathbb{C}[x]$ is principal.

$I \subseteq \mathbb{C}[x]$ ideal $\Rightarrow I = (f)$ for some f.

Let I be an ideal and pick $f \in I$ with smallest pos. degree.

possible iff $I \neq (0)$

If $I \neq \mathbb{C}[x]$ then $I = (f)$.

$g \in I$: $\quad g = q \cdot f + r$. $\quad \deg r < \deg f$ OR $r = 0$.

$\Rightarrow r \in I$.

If $r \neq 0$ then $\deg r < \deg f \Rightarrow \deg r = 0 \Rightarrow r$ const, $\neq 0$

$\Rightarrow I = \mathbb{C}[x]$. ↯

$\Rightarrow \quad I = (f)$.

Note: in $\mathbb{C}[x,y]$ there are ideals that are not principal.

(x, y)

<u>Q:</u> How many equations do I need to define a given alg. set?

<u>Hilbert's Basissatz</u> Every ideal of $\mathbb{C}[x_1, \dots, x_n]$ is finitely generated: $I = (f_1, \dots, f_m)$.

<u>Consequence:</u> $X = Z(S) = Z(\underbrace{\langle S \rangle}_{\substack{\| \\ I = (f_1, \dots, f_m)}}) = \underline{\underline{Z(f_1, \dots, f_m)}}$.

"Finitely many equations enough to define X."

<u>Hilbert's Nullstellensatz:</u> $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ ideal

$I(Z(I)) = \sqrt{I}$

Here: $\sqrt{I} = \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid f^m \in I \text{ for some } m \}$.

Cor: $Z(I) = \emptyset \iff I = \mathbb{C}[x_1, \dots, x_n]$.

Proof: $Z(I) = \emptyset \iff I(Z(I)) = \mathbb{C}[x_1, \dots, x_n]$

$\iff \sqrt{I} = \mathbb{C}[x_1, \dots, x_n]$

$\iff 1^m \in I$ for some m

$\iff 1 \in I$.

$\iff I = \mathbb{C}[x_1, \dots, x_n]$

$I = (f_1, \dots, f_m)$ $\quad Z(I) = \emptyset \iff \exists\, r_1, \dots, r_m$ s.t.

$$r_1 f_1 + r_2 f_2 + \dots + r_m f_m = 1.$$

<u>$K = \mathbb{R}$:</u> $Z(1+x^2) = \emptyset$ but $(1+x^2) \neq \mathbb{R}[x]$.

G group acting on a set X

G also acts on functions on X: $\quad g \in G$, $f : X \to \mathbb{C}$

$$(gf)(x) = f(g^{-1}x).$$

<u>Ex:</u> S_n = symmetric group in n letters acts on $\mathbb{C}^n$

$$\pi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_{\pi^{-1}(1)} \\ \vdots \\ x_{\pi^{-1}(n)} \end{pmatrix} \qquad S_n = \{ \pi : \{1, 2, \dots, n\} \circlearrowleft \}$$

Thus S_n acts on $\mathbb{C}[x_1, \dots, x_n]$.

Invariants: describe $\mathbb{C}[x_1, \dots, x_n]^{S_n}$

Ex:

$$e_1(x_1, \dots, x_n) = x_1 + \dots + x_n$$

$$e_2(x_1, \dots, x_n) = x_1 x_2 + x_1 x_3 + \dots = \sum_{1 \le i < j \le n} x_i x_j$$

$$\vdots$$

$$e_n(x_1, \dots, x_n) = x_1 \cdots x_n.$$

Every $f \in \mathbb{C}[x_1, \dots, x_n]^{S_n}$ is of the form $p(e_1, \dots, e_n)$ where p is a polynomial.

Theorem: ("Hilbert") G nice, acts on $\mathbb{C}^n$

then $\mathbb{C}[x_1,\dots,x_n]^G$ is finitely generated.

$$\mathbb{C}[x_1,\dots,x_n]^G = \{ f \in \mathbb{C}[x_1,\dots,x_n] \mid gf = f \ \forall g \in G \}$$

$X \subseteq \mathbb{C}^n$ alg. set.

$f : X \to \mathbb{C}$ is called regular if $f = F|_X$ for some $F \in \mathbb{C}[x_1,\dots,x_n]$.

$$F|_X = G|_X \iff F - G \in I(X).$$

$\mathcal{O}(X) :=$ set of all regular functions

$\mathbb{C}$- algebra.

surjective

$$\mathbb{C}[x_1, \dots, x_n] \xrightarrow[\rho]{} \mathcal{O}(X)$$

$$F \longmapsto F|_X$$

kernel of $\rho = I(X)$

<u>1st Isomorphism Thm</u>: $\mathcal{O}(X) \cong \mathbb{C}[x_1, \dots, x_n] / I(X)$.

<u>Def</u>: The pair $(X, \mathcal{O}(X))$ is called an <u>affine variety</u>.

- $X = \emptyset \iff \mathcal{O}(X) = \{0\}$.

- <u>Nullstellensatz</u>: $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ is called <u>maximal</u>

if $I \neq \mathbb{C}[x_1, \dots, x_n]$ but I is not contained in any other ideal.

The max. ideals of $\mathbb{C}[x_1, \dots, x_n]$ are precisely the ideals

$\mathfrak{m}_p \qquad p \in \mathbb{C}^n$

$$\mathfrak{m}_p = (x_1 - p_1, x_2 - p_2, \dots, x_n - p_n) = I(\{p\}).$$

Consequence: Pts of $X \xleftrightarrow{1:1}$ max ideals of $\mathbb{C}(X)$.

Introduce a topology on $\mathbb{C}^n$:

- The alg. sets in $\mathbb{C}^n$ form the closed sets of a topology namely the Zariski-Topology.

- $\emptyset, \mathbb{C}^n$ are algebraic sets.
- if $\{X_i = Z(S_i)\}_{i\in I}$ is a family of alg. sets

$$\bigcap_{i\in I} X_i = Z\left(\bigcup_i S_i\right)$$

- If $A = Z(S)$ $B = Z(T)$ are algebraic

$$A \cup B = Z(ST) \qquad ST = \{fg \mid f\in S, g\in T\}$$

"$\subseteq$" clear

"$\supseteq$" $x \in Z(ST)$, $x \notin A$ $\exists f \in S : f(x) \neq 0$.

$\forall g \in T$ $(fg)(x) = \underset{\neq 0}{f(x)}\, g(x) = 0 \Rightarrow g(x) = 0 \ \forall g \in T$

$\Rightarrow x \in B$ $\square$

$X \subseteq \mathbb{C}^n$ __closed__ if X is an alg. set.

Then every alg. set has an induced top.: $Y \subseteq X$ closed if $Y = X \cap A$ for some closed A.

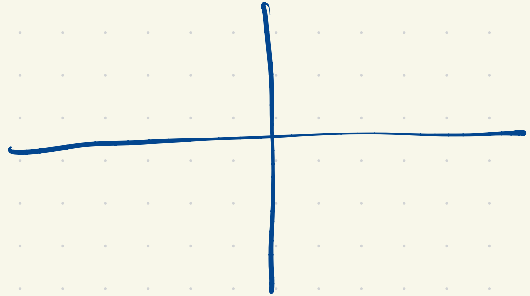
$U \subseteq \mathbb{C}^n$ __open__ if $\mathbb{C}^n \setminus U$ closed.

Polynomial functions $f : X \to \mathbb{C}$ are __continuous__.

__Def__: X, Y are sets with a topology $f : X \to Y$ continuous if $\forall A \subseteq Y$ closed $f^{-1}(A)$ is closed in X.

$X \subseteq \mathbb{C}^n$ closed: X __irreducible__ if whenever $X = A \cup B$
$\neq \emptyset$ A, B closed then $X = A$ OR $X = B$.

$$[1,2] = [1, 1/2] \cup [1/2, 2]$$

$\mathbb{C}^2$: $Z(xy) =$  $= Z(x) \cup Z(y)$

is irreducible

Fact: $X \subseteq \mathbb{C}^n$ alg-set

$$X = X_1 \cup \dots \cup X_m \qquad X_i \text{ irreducible and } X_i \not\subseteq X_j \ \forall i \neq j.$$

X_i are called irreducible components.

- $\mathbb{C}^n$ is irreducible.
- **Fact:** X irreducible $\iff$ $\mathcal{O}(X)$ is an integral domain.

($\cdot\ \mathcal{O}(X) \neq \{0\}$ and $f \cdot g = 0$ in $\mathcal{O}(X)$ then $f = 0$ OR $g = 0$)

$X = Z(xy)$ in $\mathcal{O}(X)$: $x \cdot y = 0$ not irreducible

$Z(x)$ is irreducible: $\mathcal{O}(Z(x)) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$.

$\mathcal{O}(X)$ integral domain $\iff$ $I(X)$ prime ideal.

($I(X) \neq \mathbb{C}[x_1, \dots, x_n]$
and $fg \in I(X) \Rightarrow f \in I(X)$
OR $g \in I(X)$)

(in $\mathbb{Z}$ every ideal principal:

(n) is a prime ideal $\iff$

$n \neq \pm 1$ and if $ab \in (n)$ then $a \in (n)$ or $b \in (n)$

$\iff$ $n \mid ab$ then $n \mid a$ OR $n \mid b$.

$\iff$ $\underline{n = 0}$ OR n is $\underline{\text{prime}}$.)

- X irreducible $\iff$ every open nonempty subset is $\underline{\text{dense}}$.

$U \subseteq \mathbb{C}^n$ open, $\neq \emptyset$ $\Rightarrow$ U dense in Zariski top

$\Rightarrow$ U dense in the usual top.

$$M_n(\mathbb{C}) = \mathbb{C}^{n^2}$$