Basics on Wavelet Theory and Its Applications

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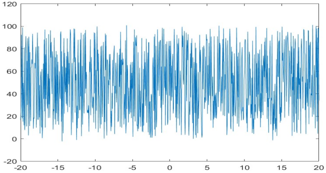
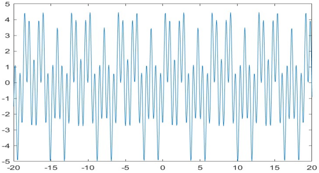
Two Popular Transforms: Fourier and Wavelet Transforms

- Two most popular transforms are the Fourier transform and the wavelet transform (and their variants).
- If you signal or data are oscillating or periodic, then the Fourier transform is often a good choice.
- If your signal or data are of multiscale nature, then the wavelet transform is often a good choice.
- Both transforms have wide applications in sciences, engineering and industry, and can be combined with other techniques such as deep neural networks.

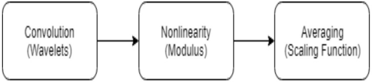




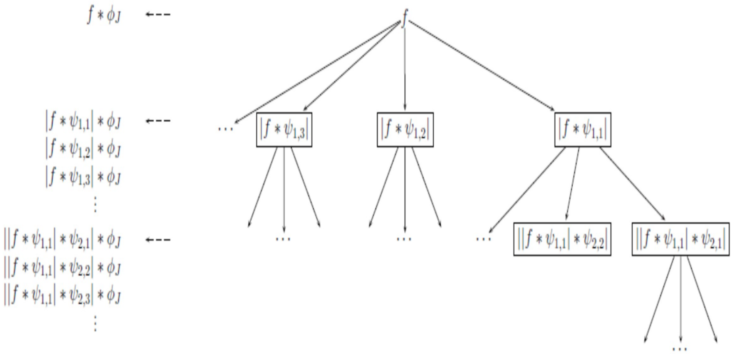
Wavelet Scattering Networks

Wavelet Scattering Transform

A wavelet scattering transform processes data in stages. The output of one stage becomes input for the next stage. Each stage consists of three operations.



The zeroth-order scattering coefficients are computed by simple averaging of the input. Here is a tree view of the algorithm:



Source: from MATLAB at https://www.mathworks.com/help/wavelet/ug/wavelet-scattering.html

Wavelets can be used in Neural Networks: MLPs and KANs

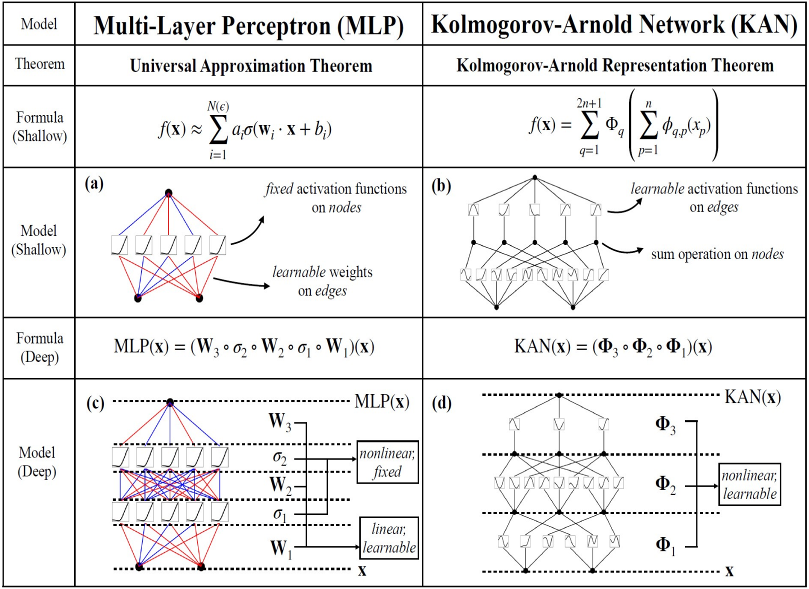


Figure 0.1: Multi-Layer Perceptrons (MLPs) vs. Kolmogorov-Arnold Networks (KANs)

Why Do We Need Transform-based Methods?

Given a particular signal to you:

$$[-21, -22, -23, -23, -25, 38, 36, 34].$$

Wavelet-based method: If you are allowed to send out only one number about this signal,

which number shall you choose?

Your answer(s):

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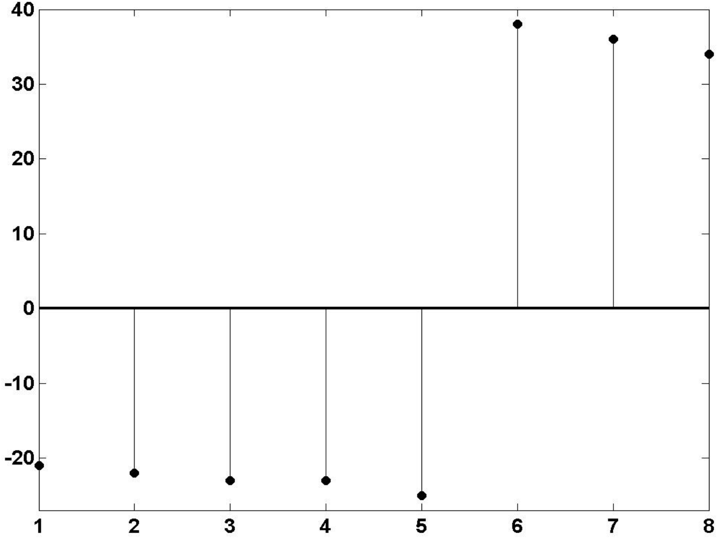
Your answer(s): **Average**

$$\frac{-21-22-23-23-25+38+36+34}{8} = -0.75.$$

Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



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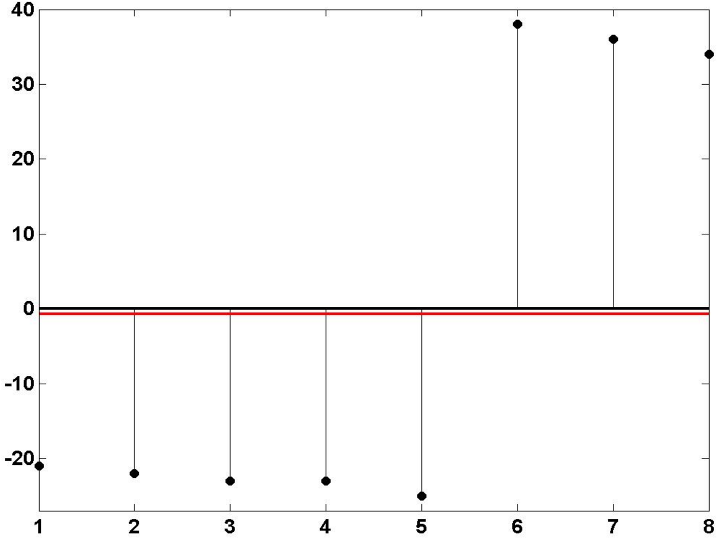

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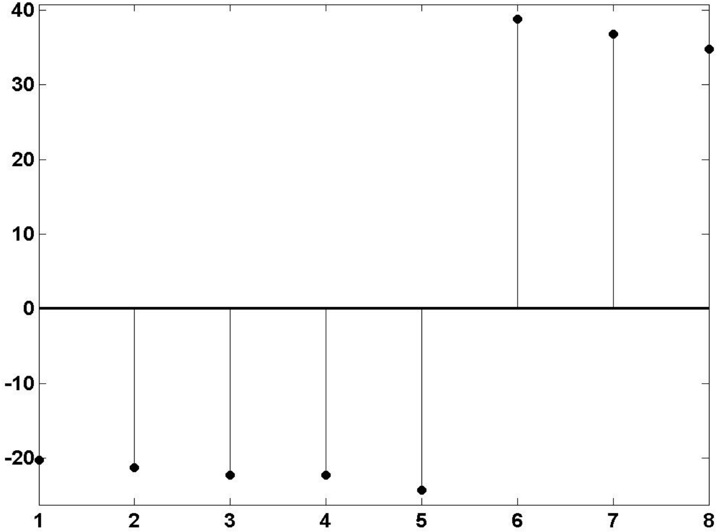
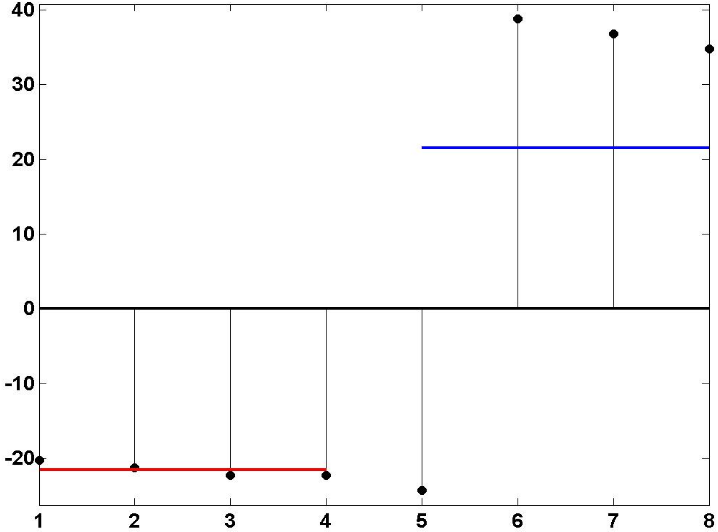


Represent $[-21, -22, -23, -23, -25, 38, 36, 34]$



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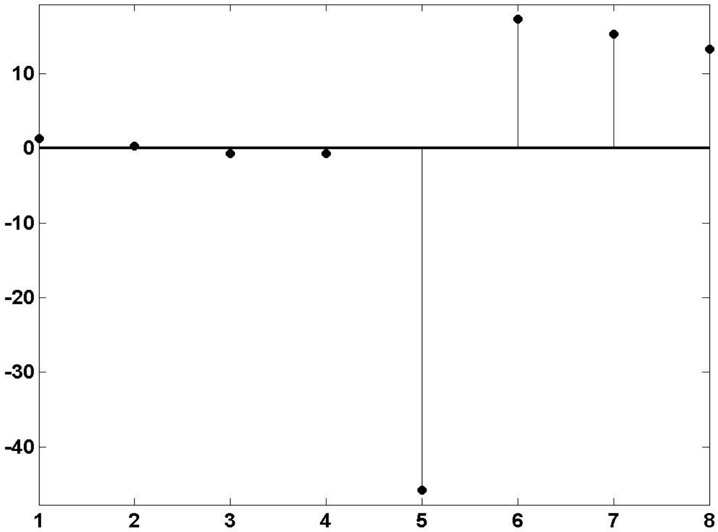

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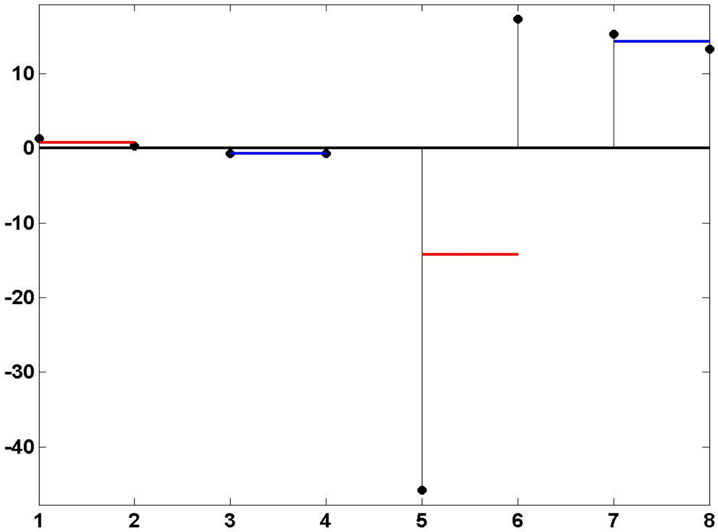

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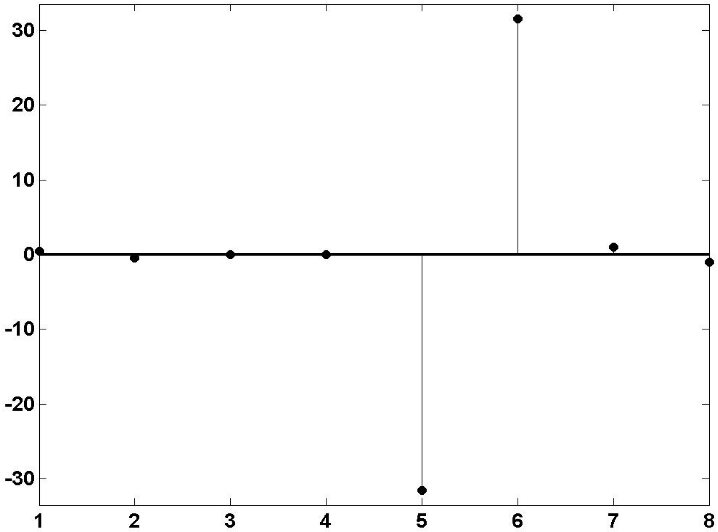
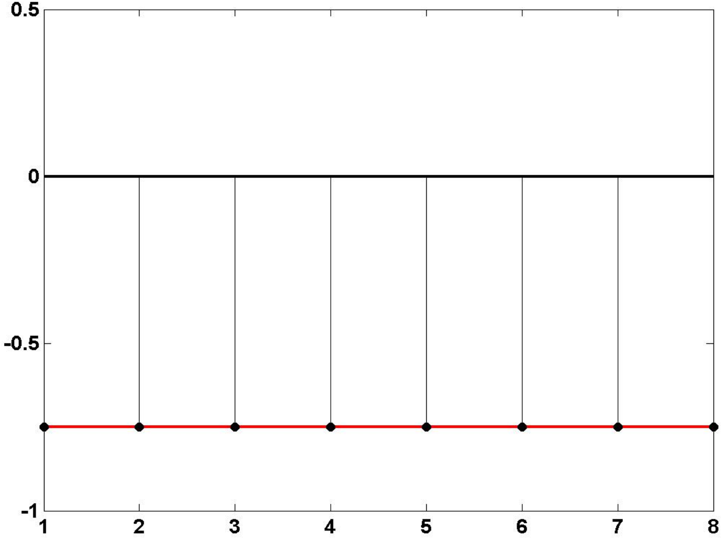


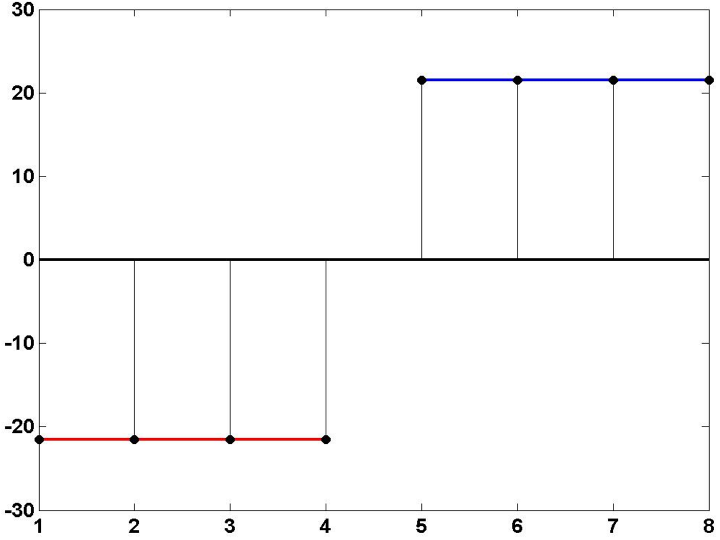
The Idea of Discrete Wavelet Transform Using Numbers

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
- Averages at level 1 (A1): -0.75,
- Average at level 2 (A2): -21.5, 21.5
- Averages at level 3 (A3): 0.75, -0.75, -14.25, 14.25.
- Averages at level 4 (A4): 0.5, -0.5, 0, 0, -31.5, 31.5, 1, -1.

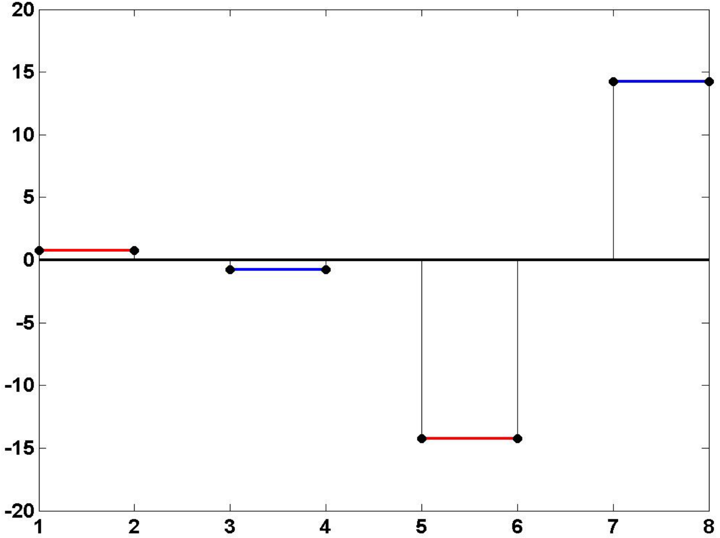
Graph of Wavelet Coefficients $A1$



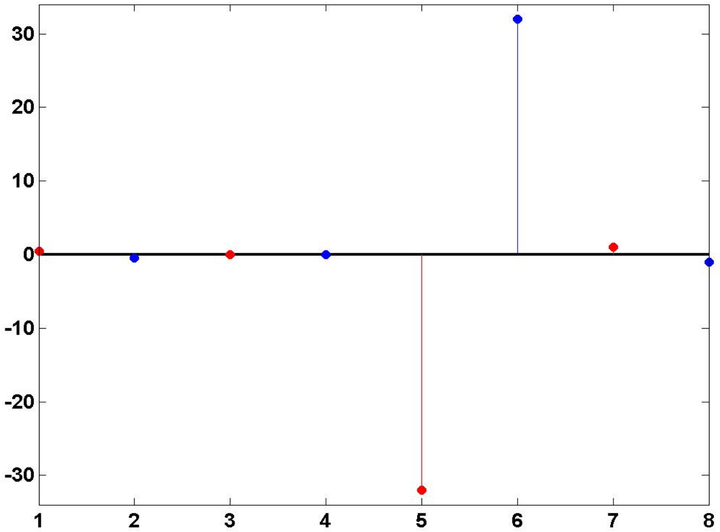
Graph of Wavelet Coefficients $A2$



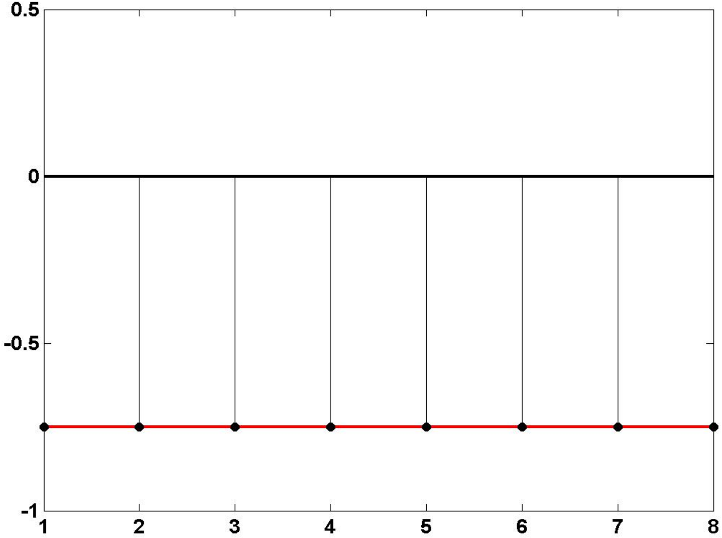
Graph of Wavelet Coefficients $A3$



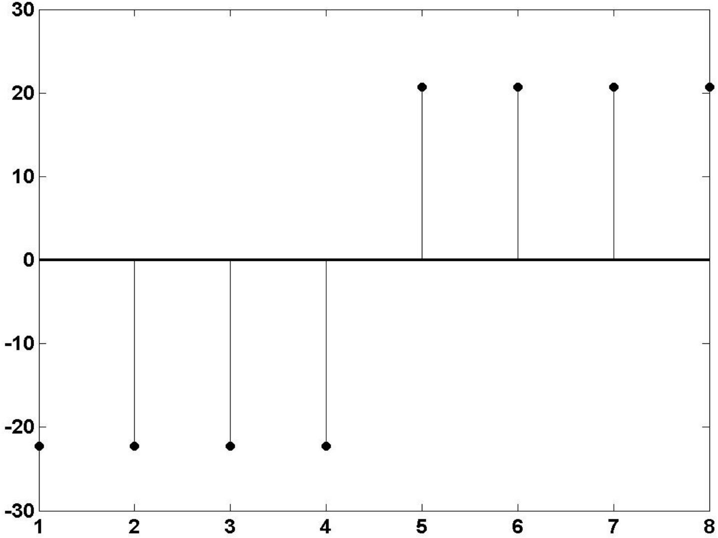
Graph of Wavelet Coefficients $A4$



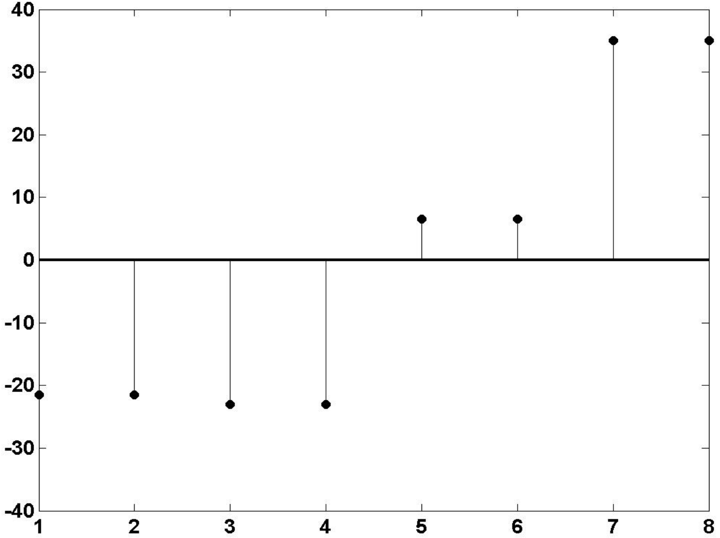
Reconstruction: $A1$ (1 number)



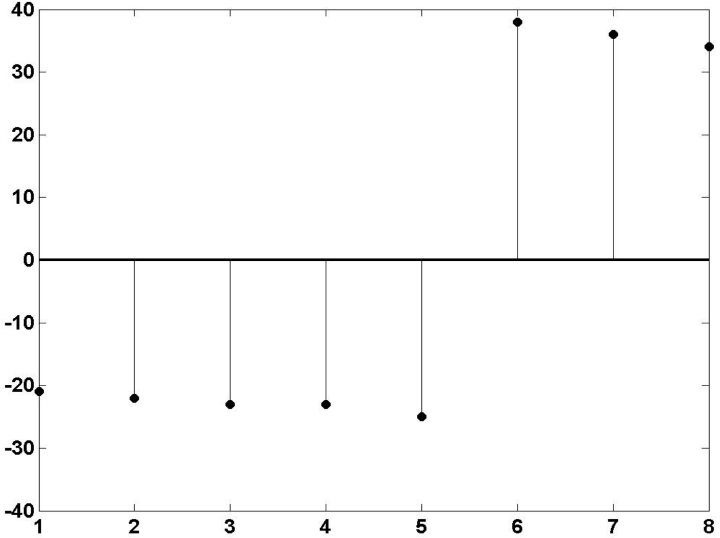
Reconstruction: $A1 + A2$ (2 numbers)



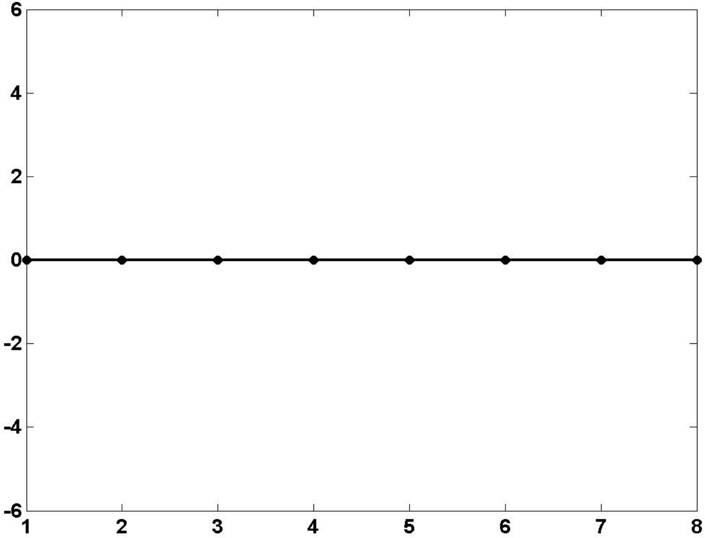
Reconstruction: $A1 + A2 + A3$ (4 numbers)



Reconstruction: $A1 + A2 + A3 + A4$ (8 numbers)



Perfect Reconstruction: Reconstructed agrees with Original



Why Are Wavelets Useful?

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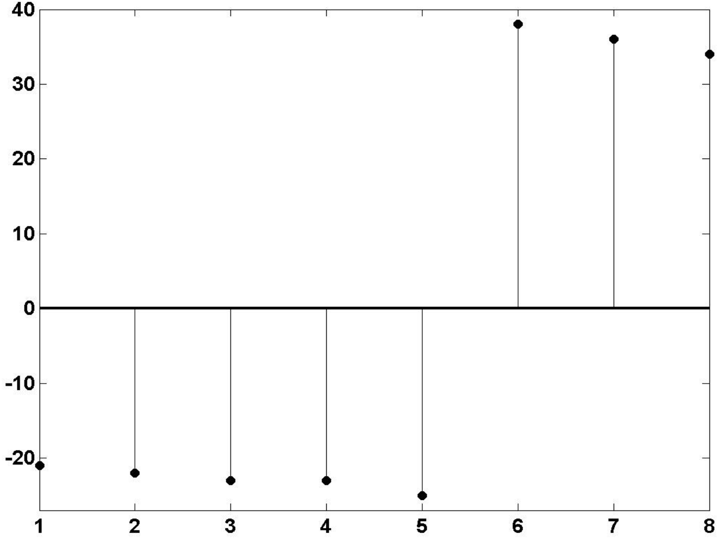
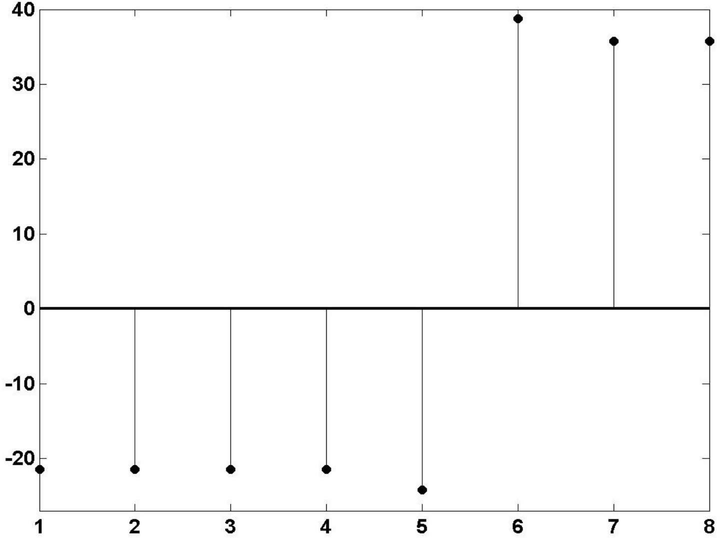
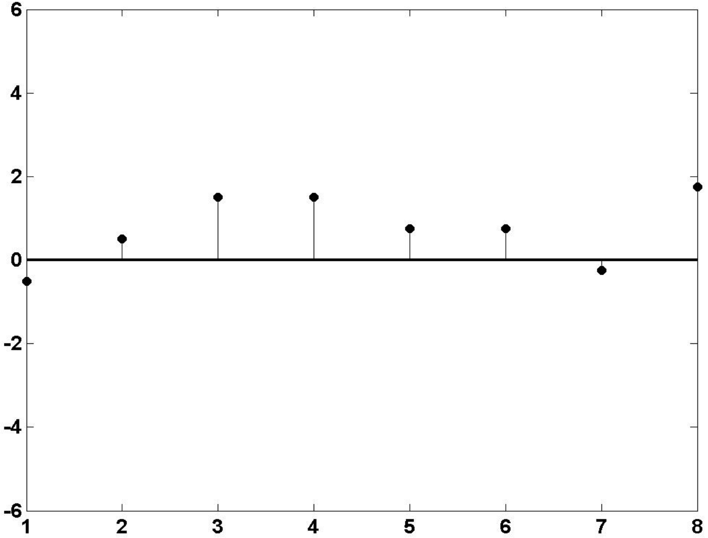
Comparison: Original



Reconstructed with 3 Numbers by Thresholding



Comparison: Original−Reconstructed



How to Compute Wavelet Coefficients Fast?

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Are we missing something for wavelets? or can we expect more from wavelets?

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For applications,
a fast computational algorithm
is highly demanded!

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.

Fast Wavelet Transform (FWT): Decomposition

- $x = [-21, -22 \mid -23, -23 \mid -25, 38 \mid 36, 34]$.
- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.

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- Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.
- Averages: $[-22.25, 20.75]$. Differences: $[0.75, -14.25]$.

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- Reconstruction: Apply subdivision scheme (prediction for doubling its size): $[-0.75] \rightarrow \underline{[-0.75, -0.75]}$.

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- Add the finest detail $\underline{[-21.5, 21.5]}$ to get $[-22.25, 20.75]$

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Problem about the Previous Fast Wavelet Transform

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$$\|x\|_{\ell_2}^2 = x_1^2 + \cdots + x_8^2 = 6504.$$

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$$\|x\|_{\ell_2}^2 = x_1^2 + \cdots + x_8^2 = 6504.$$

- But the sum of all the square of the wavelet coefficients is not 6504.
- How to make the wavelet transform have the energy preservation property?
- Without the energy preservation property, a small coefficient may carry higher energy of the signal than a large coefficient.

Fast Wavelet Transform with Energy Preservation Property

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$ with $\|x\|_{\ell_2}^2 = 6504$.
- Original Averages: $[-21.5, -23 \mid 6.5, 35]$. Difference: $[0.5, 0, -31.5, 1]$.

Fast Wavelet Transform with Energy Preservation Property

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Note that the energy preservation $3702.5 + 814.5 = 4517$ and $3702.5 + 814.5 + 1987 = 6504$.

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Note that the energy preservation $3702.5 + 814.5 = 4517$ and $3702.5 + 814.5 + 1987 = 6504$.
- Weighted Averages: $2^{3/2}[-0.75] = [2.121]$. Differences: $2^{3/2}[-21.5] = [60.80]$.
- Then $2^3 * 0.75^2 + 2^3 * 21.5^2 = 3702.5$.

Discrete Orthogonal Wavelet Transform Through Linear Algebra

- $x = [-21, -22, -23, -23, -25, 38, 36, 34]$.
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- Linear algebra interpretation: An orthonormal wavelet basis of $\mathbb{R}^8$:

$$U := \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & \frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & -\frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

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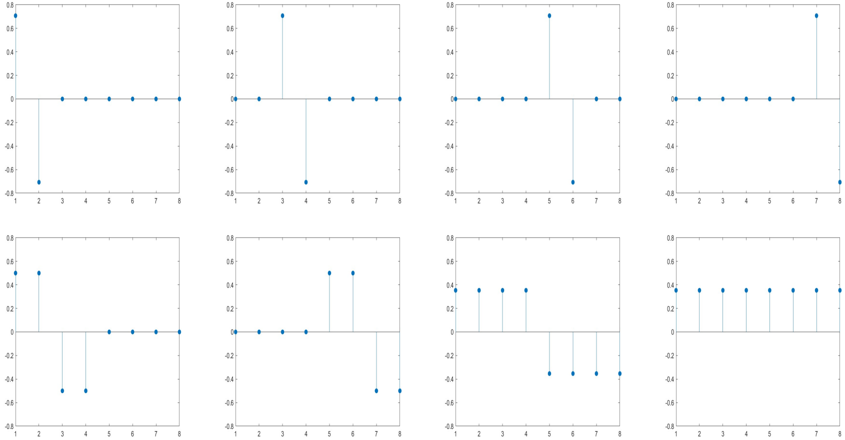
- The above orthogonal wavelet transform is generated through high-pass wavelet filter $[\frac{1}{2}, -\frac{1}{2}]$ and low-pass wavelet filter $[\frac{1}{2}, \frac{1}{2}]$ and weighted by $\sqrt{2}$.
- Perfect representation: $x = \langle x, v_1 \rangle v_1 + \cdots + \langle x, v_8 \rangle v_8$ for all $x \in \mathbb{R}^8$ with $v_1, \ldots, v_8$ being columns of the unitary matrix U, i.e., $UU^{\mathsf{T}}x = x$.

Haar Wavelet Basis Elements for $\mathbb{R}^8$



Continuous Wavelet Transform (CWT)

- **Definition:** For $f : \mathbb{R} \to \mathbb{C}$ and $1 \leqslant p \leqslant \infty$, $f \in L_p(\mathbb{R})$ if
$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{1/p} < \infty.$$
- **Definition:** The Fourier transform of a function $f \in L_1(\mathbb{R})$ is defined to be
$$(\mathscr{F} f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-ix\xi}\, dx, \qquad \xi \in \mathbb{R}.$$
- For $\psi : \mathbb{R} \to \mathbb{C}$, we define
$$\psi_{\lambda;k}(t) := |\lambda|^{1/2}\psi(\lambda t - k), \qquad \lambda \in \mathbb{R}\backslash\{0\}, k \in \mathbb{R}.$$
  Note that $\|\psi_{\lambda;k}\|_2 = \|\psi\|_2$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ and $k \in \mathbb{R}$.
- A function ψ is called an admissible wavelet if
$$C_\psi := \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty.$$
- For $f, \psi \in L_2(\mathbb{R})$ such that ψ is an admissible wavelet, the continuous wavelet transform (CWT) of the function f is defined to be
$$\mathcal{W}_\psi f(\alpha, \beta) := \langle f, \psi_{\alpha^{-1};\alpha^{-1}\beta} \rangle = \int_{\mathbb{R}} f(x) |\alpha|^{-1/2} \overline{\psi\left(\frac{x-\beta}{\alpha}\right)} dx,$$
$$\alpha \in \mathbb{R}\backslash\{0\}, \beta \in \mathbb{R}.$$

Scales and Translations of Wavelet Functions

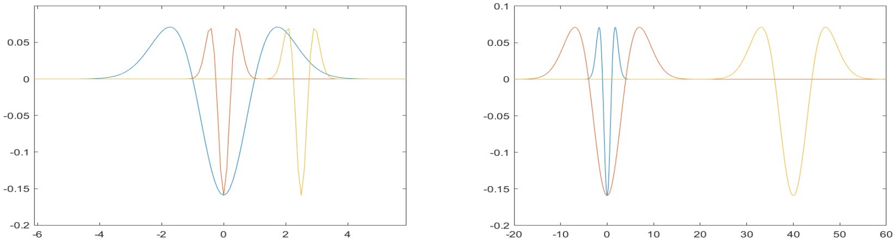
- A typical example of admissible wavelets is given by

$$\psi(x) = G''(t) = \frac{1}{2\pi}(t^2 - 1)e^{-t^2/2}, \quad \text{where} \quad G(t) = \frac{1}{2\pi}e^{-t^2/2}.$$

- Recall that $\psi_{\lambda;k}(t) := |\lambda|^{1/2}\psi(\lambda t - k)$. A discretized version of continuous wavelet transform leads a classical wavelet system:

$$\psi_{2^j;k} = 2^{j/2}\psi(2^j \cdot -k), \qquad j \in \mathbb{Z}, k \in \mathbb{Z}.$$

- Note that $\langle f, \psi_{2^j;k}\rangle = \mathcal{W}_\psi f(2^{-j}, 2^{-j}k)$ due to $\psi_{\alpha^{-1};\alpha^{-1}\beta} = \psi_{2^j;k}$ with $\alpha = 2^{-j}$ and $\beta = 2^{-j}\beta$.



ψ (blue), $\psi_{2^2;0}$ and $\psi_{2^2;10}$ (orange, left), and $\psi_{2^{-2};0}$ and $\psi_{2^{-2};10}$ (orange, right).

Key Property of CWT

Theorem

Let $\psi, \eta \in L_2(\mathbb{R})$ *such that*

$$C_\psi := \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi < \infty, \qquad C_\eta := \int_{\mathbb{R}} \frac{|\widehat{\eta}(\xi)|^2}{|\xi|} d\xi < \infty.$$

Then $\|\mathcal{W}_\psi f\|^2 := \langle \mathcal{W}_\psi f, \mathcal{W}_\psi f\rangle = C_\psi \|f\|^2_{L_2(\mathbb{R})}$ *and for all* $f, g \in L_2(\mathbb{R})$,

$$\langle \mathcal{W}_\psi f, \mathcal{W}_\eta g\rangle := \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_\psi f(\alpha, \beta) \overline{\mathcal{W}_\eta g(\alpha, \beta)} d\beta \frac{d\alpha}{\alpha^2} = C_{\psi,\eta} \langle f, g\rangle,$$

where $C_{\psi,\eta} := \int_{\mathbb{R}} \frac{\overline{\widehat{\psi}(\xi)}\widehat{\eta}(\xi)}{|\xi|} d\xi < \infty$. *In particular, if* $C_{\psi,\eta} \neq 0$, *then*

$$f(\cdot) = \frac{1}{C_{\psi,\eta}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_\psi f(\alpha, \beta) \eta_{\alpha^{-1};\alpha^{-1}\beta}(\cdot) d\beta \frac{d\alpha}{\alpha^2}$$

holds in the weak sense, that is,

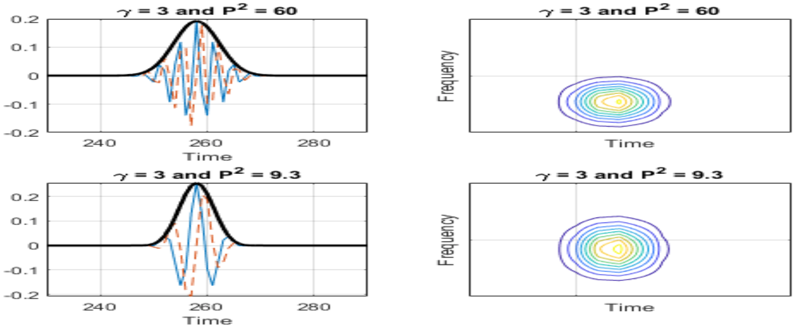
$$\langle f, g\rangle = \frac{1}{C_{\psi,\eta}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{W}_\psi f(\alpha, \beta) \langle \eta_{\alpha^{-1};\alpha^{-1}\beta}, g\rangle d\beta \frac{d\alpha}{\alpha^2}, \qquad \forall g \in L_2(\mathbb{R}).$$

Admissible Wavelets

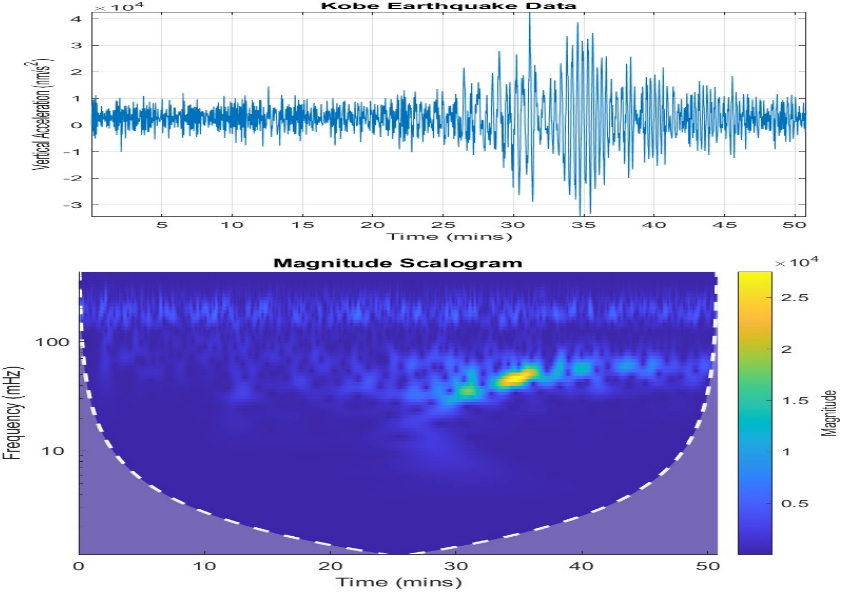
- $\psi(x) = G''(t) = \frac{1}{2\pi}(t^2 - 1)e^{-t^2/2}$, where $G(t) = \frac{1}{2\pi}e^{-t^2/2}$.
- Morlet (or Gabor) wavelets: $\psi(t) = c_\sigma \pi^{-1/4} e^{-t^2/2}(e^{i\sigma t} - e^{-\sigma^2/2})$ with $c_\sigma = (1 + e^{-\sigma^2} - 2e^{-\frac{3}{4}\sigma^2})^{-\frac{1}{2}}$ and $\widehat{\psi}(\xi) = c_\sigma \pi^{-\frac{1}{4}}(e^{-(\sigma-\xi)^2/2} - e^{-(\xi^2+\sigma^2)/2})$.
- Analytic wavelets ψ are often used such that $\widehat{\psi}(\xi) = 0$ for all $\xi < 0$.
- The generalized Morse wavelet is given by

$$\widehat{\psi}(\xi) := a_{P,\gamma}\chi_{[0,1]}(\xi)\xi^{\frac{P^2}{\gamma}} e^{-\xi^\gamma},$$

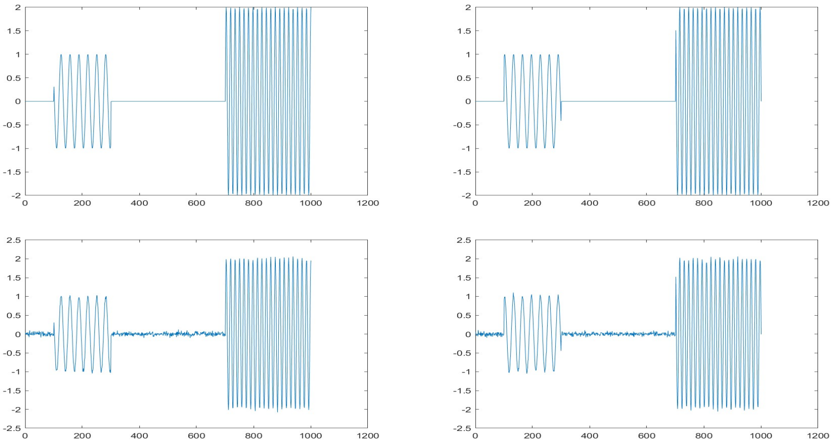
  where $a_{\beta,\gamma}$ is a normalizing constant and γ characterizes the symmetry of the Morse wavelet. The Morse wavelet is obtained by replacing P^2/γ with β.



Scalogram Using CWT



$f(t) = e^{2\pi i 32 t}\chi_{[0.1,0.3]} + 2e^{-2\pi i 64 t}\chi_{(0.7,\infty)}$



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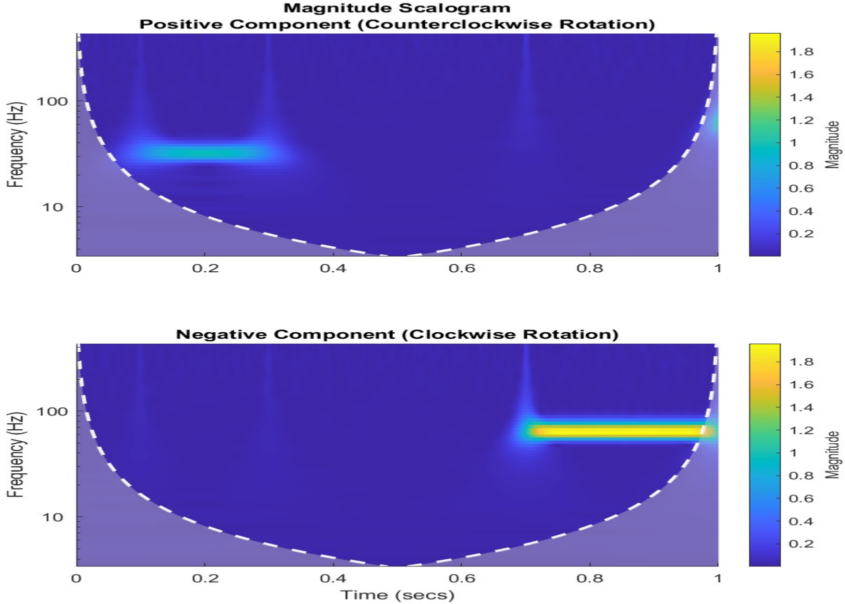

Discrete Framelet Transform (DFrT) and Discrete Wavelet Transform (DWT)

- One-level discrete framelet transform
- Discrete framelet transform vs discrete wavelet transform
- Basic properties of DFrT: perfect reconstruction and sparsity.
- Different types of wavelets and framelets.

In this course, we only consider real-valued filters, sequences and data.

Some Definitions and Notation

- $l(\mathbb{Z})$ consists of all real-valued sequences $v = \{v[k]\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{R}$.
- $l_2(\mathbb{Z})$ for signals: all real-valued sequences $v \in l_2(\mathbb{Z})$ such that

$$\|v\|_{\ell_2}^2 := \sum_{k\in\mathbb{Z}} |v[k]|^2 < \infty.$$

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$$\mathsf{v}(z) := \sum_{k\in\mathbb{Z}} v[k] z^k, \qquad z \in \mathbb{C}\backslash\{0\},$$

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Subdivision and Transition Operators

- The subdivision operator $\mathcal{S}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$:

$$(\mathcal{S}_b v)[n] := 2 \sum_{k \in \mathbb{Z}} v[k] b[n - 2k], \quad n \in \mathbb{Z}$$

Often used in the reconstruction step in a fast wavelet transform.

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$$\mathcal{S}_b v = 2b * (v\uparrow 2) \quad \text{and} \quad \mathcal{T}_b v = 2(b^\star * v)\downarrow 2.$$

- If the filter $b \in l_0(\mathbb{Z})$ has short support, then the length of $\mathcal{S}_b v$ is almost twice of that of v, while the length of $\mathcal{T}_b v$ is only half of that of v.

One-level Discrete Framelet Transform (DFrT)

- Let $\tilde{b}_0, \ldots, \tilde{b}_s, b_0, \ldots, b_s \in l_0(\mathbb{Z})$ be finitely supported filters in $l_0(\mathbb{Z})$.
- For input data $v \in l_2(\mathbb{Z})$, a one-level discrete framelet decomposition:

$$w_\ell := \tfrac{\sqrt{2}}{2}\, \mathcal{T}_{\tilde{b}_\ell} v = \sqrt{2}(\tilde{b}_\ell^\star * v)\downarrow 2, \qquad \ell = 0, \ldots, s,$$

  or using a framelet decomposition operator:

$$\tilde{\mathcal{W}} v := \tfrac{\sqrt{2}}{2}(\mathcal{T}_{\tilde{b}_0} v, \ldots, \mathcal{T}_{\tilde{b}_s} v) = \sqrt{2}(\tilde{b}_0^\star * v, \ldots, \tilde{b}_s^\star * v)\downarrow 2.$$

- The total number of nonzero coefficients of w_ℓ is approximately half of that of v. Hence, $\tilde{\mathcal{W}} v$ is roughly $\frac{1}{2}(s+1)$ of that of v.

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- A one-level framelet reconstruction by $\mathcal{V} : (l(\mathbb{Z}))^{1\times(s+1)} \to l(\mathbb{Z})$:

$$\mathcal{V}(w_0, \dots, w_s) = \frac{\sqrt{2}}{2} \sum_{\ell=0}^{s} \mathcal{S}_{b_\ell} w_\ell = \sqrt{2} b_0 * (w_0 \uparrow 2) + \cdots + \sqrt{2} b_s * (w_s \uparrow 2).$$

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- A filter bank $(\{\tilde{b}_0, \dots \tilde{b}_s\}, \{b_0, \dots, b_s\})$ has the perfect reconstruction (PR) if $\mathcal{V}\tilde{\mathcal{W}} v = v$ for all data $v \in l_2(\mathbb{Z})$.

Diagram of One-level DFrTs

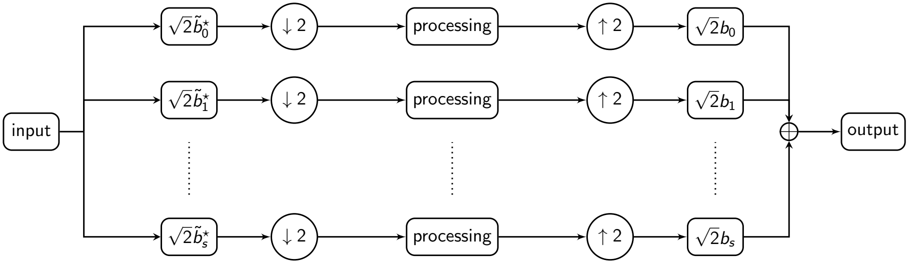

Figure: Diagram of a one-level discrete framelet transform using a pair of filter banks $(\{\tilde{b}_0, \dots, \tilde{b}_s\}, \{b_0, \dots, b_s\})$. Decomposition: $w_\ell := \sqrt{2}(\tilde{b}_\ell^\star * v)\downarrow 2$ for $\ell = 0, \dots, s$. Reconstruction: $\sqrt{2}b_0 * (w_0\uparrow 2) + \cdots + \sqrt{2}b_s * (w_s\uparrow 2)$.

It is called a wavelet filter bank for $s = 1$, and a framelet filter bank if $s > 1$.

Property of DFrT: Perfect Reconstruction (PR) Property

Theorem (PR of DFrT)

A pair of filter banks $(\{\tilde{b}_0, \ldots, \tilde{b}_s\}, \{b_0, \ldots, b_s\})$ *has*

$$\textit{Perfect Reconstruction (PR):} \quad v = \mathcal{V}\tilde{\mathcal{W}}v = \frac{1}{2}\sum_{\ell=0}^{s} \mathcal{S}_{b_\ell}\mathcal{T}_{\tilde{b}_\ell}v, \;\; \forall\, v \in l_2(\mathbb{Z}),$$

if and only if $(\{\tilde{b}_0, \ldots, \tilde{b}_s\}, \{b_0, \ldots, b_s\})$ *is a dual framelet filter bank* satisfying:

$$\tilde{\mathrm{b}}_0(z)\mathrm{b}_0(z^{-1}) + \tilde{\mathrm{b}}_1(z)\mathrm{b}_1(z^{-1}) + \cdots + \tilde{\mathrm{b}}_s(z)\mathrm{b}_s(z^{-1}) = 1, \quad (1)$$

$$\tilde{\mathrm{b}}_0(z)\mathrm{b}_0(-z^{-1}) + \tilde{\mathrm{b}}_1(z)\mathrm{b}_1(-z^{-1}) + \cdots + \tilde{\mathrm{b}}_s(z)\mathrm{b}_s(-z^{-1}) = 0. \quad (2)$$

That is,

$$\begin{bmatrix} \tilde{\mathrm{b}}_0(z) & \cdots & \tilde{\mathrm{b}}_s(z) \\ \tilde{\mathrm{b}}_0(-z) & \cdots & \tilde{\mathrm{b}}_s(-z) \end{bmatrix} \begin{bmatrix} \mathrm{b}_0(z) & \cdots & \mathrm{b}_s(z) \\ \mathrm{b}_0(-z) & \cdots & \mathrm{b}_s(-z) \end{bmatrix}^\star = I_2.$$

where I_2 *denotes the* 2×2 *identity matrix and* $\mathrm{A}^\star(z) := [\mathrm{A}(z^{-1})]^{\mathsf{T}}$ *for a general matrix A.*

Biorthogonal Wavelet Filter Banks

Definition: A dual framelet filter bank with $s = 1$ is called a biorthogonal wavelet filter bank, a nonredundant filter bank. That is, $(\{\tilde{b}_0, \tilde{b}_1\}, \{b_0, b_1\})$ is a biorthogonal wavelet filter bank if

$$\begin{bmatrix} \tilde{b}_0(z) & \tilde{b}_1(z) \\ \tilde{b}_0(-z) & \tilde{b}_1(-z) \end{bmatrix} \begin{bmatrix} b_0(z) & b_1(z) \\ b_0(-z) & b_1(-z) \end{bmatrix}^\star = I_2.$$

Moreover, the above identity is equivalent to

$$\begin{bmatrix} b_0(z) & b_1(z) \\ b_0(-z) & b_1(-z) \end{bmatrix}^\star \begin{bmatrix} \tilde{b}_0(z) & \tilde{b}_1(z) \\ \tilde{b}_0(-z) & \tilde{b}_1(-z) \end{bmatrix} = I_2,$$

which is just

$$\begin{bmatrix} b_0(z^{-1}) & b_0(-z^{-1}) \\ b_1(z^{-1}) & b_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} \tilde{b}_0(z) & \tilde{b}_1(z) \\ \tilde{b}_0(-z) & \tilde{b}_1(-z) \end{bmatrix} = I_2,$$

that is,

$$\tilde{b}_0(z)b_0(z^{-1}) + \tilde{b}_0(-z)b_0(-z^{-1}) = 1, \qquad \tilde{b}_1(z)b_1(z^{-1}) + \tilde{b}_1(-z)b_1(-z^{-1}) = 1,$$

$$\tilde{b}_0(z)b_1(z^{-1}) + \tilde{b}_0(-z)b_1(-z^{-1}) = 0, \qquad \tilde{b}_1(z)b_0(z^{-1}) + \tilde{b}_1(-z)b_0(-z^{-1}) = 0.$$

Orthogonal Wavelet Filter Banks

Definition: $\{b_0, b_1\}$ is called an orthogonal wavelet filter bank if $(\{b_0, b_1\}, \{b_0, b_1\})$ is a dual framelet filter bank, that is,

$$\begin{bmatrix} \mathrm{b}_0(z) & \mathrm{b}_1(z) \\ \mathrm{b}_0(-z) & \mathrm{b}_1(-z) \end{bmatrix} \begin{bmatrix} \mathrm{b}_0(z) & \mathrm{b}_1(z) \\ \mathrm{b}_0(-z) & \mathrm{b}_1(-z) \end{bmatrix}^{\star} = I_2,$$

which is further equivalent to

$$\begin{bmatrix} \mathrm{b}_0(z^{-1}) & \mathrm{b}_0(-z^{-1}) \\ \mathrm{b}_1(z^{-1}) & \mathrm{b}_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} \mathrm{b}_0(z) & \mathrm{b}_1(z) \\ \mathrm{b}_0(-z) & \mathrm{b}_1(-z) \end{bmatrix} = I_2,$$

which is further equivalent to

$$\begin{aligned} \mathrm{b}_0(z)\mathrm{b}_0(z^{-1}) + \mathrm{b}_0(-z)\mathrm{b}_0(-z^{-1}) &= 1, \\ \mathrm{b}_1(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_1(-z)\mathrm{b}_1(-z^{-1}) &= 1, \\ \mathrm{b}_0(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_0(-z)\mathrm{b}_1(-z^{-1}) &= 0. \end{aligned}$$

That is, the vectors

$$\langle \mathrm{b}_0(z), \mathrm{b}_0(-z) \rangle \quad \text{and} \quad \langle \mathrm{b}_1(z), \mathrm{b}_1(-z) \rangle$$

have ℓ_2-norm one for all $z \in \mathbb{C}\backslash\{0\}$ and they are mutually perpendicular in $\mathbb{C}^2$.

Role of $\frac{\sqrt{2}}{2}$ in DFrT: Norm Preservation Property

Theorem

Let $b_0, \ldots, b_s \in l_0(\mathbb{Z})$. Then the following are equivalent to each other:

(i) $\langle \mathcal{W}v, \mathcal{W}\tilde{v} \rangle = \langle v, \tilde{v} \rangle$ *for all* $v, \tilde{v} \in l_2(\mathbb{Z})$, *where*

$$\mathcal{W}v := \tfrac{\sqrt{2}}{2}(\mathcal{T}_{b_0}v, \ldots, \mathcal{T}_{b_s}v) = \sqrt{2}(b_0^\star * v, \ldots, b_s^\star * v)\downarrow 2.$$

(ii) $\|\mathcal{W}v\|^2_{(l_2(\mathbb{Z}))^{1\times(s+1)}} = \|v\|^2_{l_2(\mathbb{Z})}$ *for all* $v \in l_2(\mathbb{Z})$, *that is,*

$$\|\tfrac{\sqrt{2}}{2}\mathcal{T}_{b_0}v\|^2_{l_2(\mathbb{Z})} + \cdots + \|\tfrac{\sqrt{2}}{2}\mathcal{T}_{b_s}v\|^2_{l_2(\mathbb{Z})} = \|v\|^2_{l_2(\mathbb{Z})}, \quad \forall\, v \in l_2(\mathbb{Z}).$$

(iii) *The filter bank* $(\{b_0, \ldots, b_s\}, \{b_0, \ldots, b_s\})$ *has PR:*

$$\begin{bmatrix} \mathrm{b}_0(z) & \cdots & \mathrm{b}_s(z) \\ \mathrm{b}_0(-z) & \cdots & \mathrm{b}_s(-z) \end{bmatrix} \begin{bmatrix} \mathrm{b}_0(z) & \cdots & \mathrm{b}_s(z) \\ \mathrm{b}_0(-z) & \cdots & \mathrm{b}_s(-z) \end{bmatrix}^\star = I_2.$$

Definition: $\{b_0, \ldots, b_s\}$ with PR is called a tight framelet filter bank. If in addition $s = 1$, it is called an orthogonal wavelet filter bank.

Examples of Orthogonal Wavelet Filter Banks

To list a filter $b = \{b[k]\}_{k \in \mathbb{Z}}$ with support $[m, n]$,

$$b = \{b[m], \ldots, b[-1], \underline{\mathbf{b[0]}}, b[1], \ldots, b[n]\}_{[m,n]},$$

- $\{b_0, b_1\}$ is the Haar orthogonal wavelet filter bank, where

$$b_0 = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]}, \qquad b_1 = \{\underline{-\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]}.$$

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- Note $\mathrm{b}_0(z) = \frac{1}{2}(1 + z)$, $\mathrm{b}_1(z) = \frac{1}{2}(z - 1)$. Hence, $\{b_0, b_1\}$ satisfies

$$\begin{aligned}
\mathrm{b}_0(z)\mathrm{b}_0(z^{-1}) + \mathrm{b}_0(-z)\mathrm{b}_0(-z^{-1}) &= 1,\\
\mathrm{b}_1(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_1(-z)\mathrm{b}_1(-z^{-1}) &= 1,\\
\mathrm{b}_0(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_0(-z)\mathrm{b}_1(-z^{-1}) &= 0.
\end{aligned}$$

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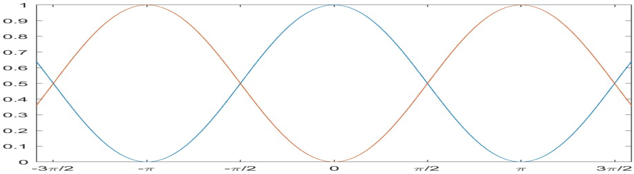
\mathrm{b}_0(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_0(-z)\mathrm{b}_1(-z^{-1}) &= 0.
\end{aligned}$$

- The graphs of $|\mathrm{b}_0(e^{-i\xi})|^2 = \cos^2(\xi/2)$ (blue) and $|\mathrm{b}_1(e^{-i\xi})|^2 = \sin^2(\xi/2)$ (red):



Examples of Biorthogonal Wavelet Filter Banks

- $(\{\tilde{b}_0, \tilde{b}_1\}, \{b_0, b_1\})$ is a biorthogonal wavelet filter bank, where

$$\tilde{b}_0 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}, \qquad \tilde{b}_1 = \{\underline{-\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, -\tfrac{1}{4}\}_{[0,2]},$$

$$b_0 = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad b_1 = \{-\tfrac{1}{8}, \underline{-\mathbf{\tfrac{1}{4}}}, \tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{8}\}_{[-1,3]}.$$

Called the LeGall wavelet filter bank, used for image compression.

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  Called the LeGall wavelet filter bank, used for image compression.

- Note that $\tilde{\mathrm{b}}_0(z) = \frac{1}{8}(1+z)^2(-z^(-2) + 4z^{-1} - 1)$, $\tilde{\mathrm{b}}_1(z) = -\frac{1}{4}z^{-1}(1-z)^2$, $\mathrm{b}_0(z) = \frac{1}{4}z^{-1}(1+z)^2$, and $\mathrm{b}_1(z) = \frac{1}{8}(1-z)^2(-z^{-1} - 4 - z)$, satisfying

$$\begin{bmatrix} \mathrm{b}_0(z^{-1}) & \mathrm{b}_0(-z^{-1}) \\ \mathrm{b}_1(z^{-1}) & \mathrm{b}_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} \tilde{b}_0(z) & \tilde{b}_1(z) \\ \tilde{b}_0(-z) & \tilde{b}_1(-z) \end{bmatrix} = I_2,$$

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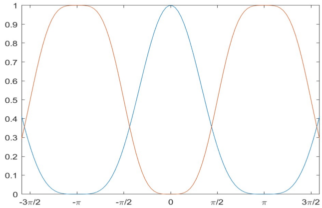
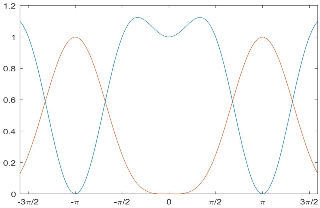
$$\begin{bmatrix} \mathrm{b}_0(z^{-1}) & \mathrm{b}_0(-z^{-1}) \\ \mathrm{b}_1(z^{-1}) & \mathrm{b}_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} \tilde{b}_0(z) & \tilde{b}_1(z) \\ \tilde{b}_0(-z) & \tilde{b}_1(-z) \end{bmatrix} = I_2,$$

- Frequency responses of $|\mathrm{b}_0(e^{-i\xi})|^2, |\mathrm{b}_1(e^{-i\xi})|^2$ and $|\tilde{\mathrm{b}}_0(e^{-i\xi})|^2, |\tilde{b}_1(e^{-i\xi})|^2$.





Discrete Wavelet Transform Using Haar Wavelet Filter Bank

Apply the Haar orthogonal wavelet filter bank to

$$v = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]}$$

Note that $\mathcal{T}_{b_0} v[n] = v[2n] + v[2n+1]$ and $\mathcal{T}_{b_1} v[n] = -v[2n] + v[2n+1]$. We have the wavelet coefficients:

$$w_0 = \frac{\sqrt{2}}{2}\{-43, -46, 13, 70\}_{[0,3]}, \quad w_1 = \frac{\sqrt{2}}{2}\{-1, 0, 63, -2\}_{[0,3]}.$$

Note that

$$(\mathcal{S}_{b_0} w_0)[2n] = w_0[n], \quad (\mathcal{S}_{b_0} w_0)(2n+1) = w_0[n], \quad n \in \mathbb{Z}$$
$$(\mathcal{S}_{b_1} w_1)(2n) = -w_1[n], \quad (\mathcal{S}_{b_1} w_1)(2n+1) = w_1[n], \quad n \in \mathbb{Z}.$$

Hence, we have

$$\frac{\sqrt{2}}{2}\mathcal{S}_{b_0} w_0 = \frac{1}{2}\{-43, -43, -46, -46, 13, 13, 70, 70\}_{[0,7]},$$
$$\frac{\sqrt{2}}{2}\mathcal{S}_{b_1} w_1 = \frac{1}{2}\{1, -1, 0, 0, -63, 63, 2, -2\}_{[0,7]}.$$

Clearly, we have the perfect reconstruction of v:

$$\frac{\sqrt{2}}{2}\mathcal{S}_{b_0} w_0 + \frac{\sqrt{2}}{2}\mathcal{S}_{b_1} w_1 = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]} = v$$

and the following energy-preserving identity

$$\|w_0\|^2_{l_2(\mathbb{Z})} + \|w_1\|^2_{l_2(\mathbb{Z})} = 4517 + 1987 = 6504 = \|v\|^2_{l_2(\mathbb{Z})}.$$

Property of DFrT: Sparsity

- One key feature of DFrT is its sparse representation for smooth or piecewise smooth signals.
- It is desirable to have as many as possible negligible framelet coefficients for smooth signals.
- Smooth signals are modeled by polynomials. Let $\mathsf{p} : \mathbb{R} \to \mathbb{C}$ be a polynomial:

$$\mathsf{p}(x) = \sum_{n=0}^{m} p_n x^n.$$

- a polynomial sequence $\mathsf{p}|_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ by

$$(\mathsf{p}|_{\mathbb{Z}})[k] = \mathsf{p}(k), \qquad k \in \mathbb{Z}.$$

- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- Π_{m-1} is the set of all polynomials of degree less than m.

Transition Operator Acting on Polynomials

Theorem

Let $b = \{b[k]\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$. *Then for any* $m \in \mathbb{N}$, *the following are equivalent:*

1. $\mathcal{T}_b \mathrm{p} = 0$ *for all polynomial sequences* $\mathrm{p} \in \Pi_{m-1}$, *where* $(\mathcal{T}_b \mathrm{p})[n] := 2\sum_{k\in\mathbb{Z}} \mathrm{p}[k] b[k-2n]$.
2. *The filter* b *must have* *m vanishing moments*:
$$\sum_{k\in\mathbb{Z}} b[k]k^j = 0, \qquad j = 0, \ldots, m-1.$$
3. $\mathrm{b}(1) = \mathrm{b}'(1) = \cdots = \mathrm{b}^{(m-1)}(1) = 0$.
4. $(1-z)^m \mid \mathrm{b}(z)$, *i.e.*, $\mathrm{b}(z) = (1-z)^m Q(z)$ *for some Laurent polynomial* Q.

Subdivision Operator Acting on Polynomials

Theorem

Let $b = \{b[k]\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$. *For* $m \in \mathbb{N}$, *the following are equivalent:*

1. $\mathcal{S}_b\Pi_{m-1} \subseteq \Pi_{m-1}$, *where* $(\mathcal{S}_b\mathrm{p})[n] = 2\sum_{k\in\mathbb{Z}} \mathrm{p}[k]b[n-2k]$.
2. b *has order* m *sum rules:*

$$\sum_{k\in\mathbb{Z}} b[2k](2k)^j = \sum_{k\in\mathbb{Z}} b[1+2k](1+2k)^j, \quad j = 0, \ldots, m-1.$$

3. b *has order* m *sum rules:*

$$\mathrm{b}(-1) = \mathrm{b}'(-1) = \cdots = \mathrm{b}^{(m-1)}(-1) = 0.$$

4. $(1+z)^m \mid \mathrm{b}(z)$, *i.e.,* $\mathrm{b}(z) = (1+z)^m Q(z)$ *for some Laurent polynomial* Q.

Moreover, if any of the above holds, then for all $\mathrm{p} \in \Pi_{m-1}$,

$$\mathcal{S}_b\mathrm{p} = 2^{-1}\mathrm{p}(2^{-1}\cdot) * b = \sum_{j=0}^{\infty} \frac{(-i)^j}{2^j j!} \mathrm{p}^{(j)}(2^{-1}\cdot)[\mathrm{b}(e^{-i\xi})]^{(j)}|_{\xi=0}.$$

Equivalences among (2), (3), (4) are easy. So, we only prove (1) $\Longleftrightarrow$ (2).

Vanishing Moments for Biorthogonal Wavelet Filter Banks

Theorem

Let $(\{\tilde{b}_0, \tilde{b}_1\}, \{b_0, b_1\})$ *be a biorthogonal wavelet filter bank, i.e.,*

$$\begin{bmatrix} \mathrm{b}_0(z^{-1}) & \mathrm{b}_0(-z^{-1}) \\ \mathrm{b}_1(z^{-1}) & \mathrm{b}_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} \tilde{\mathrm{b}}_0(z) & \tilde{\mathrm{b}}_1(z) \\ \tilde{\mathrm{b}}_0(-z) & \tilde{\mathrm{b}}_1(-z) \end{bmatrix} = I_2.$$

Then $\tilde{b}_1$ *has* m *vanishing moments if and only if* b_0 *has* m *sum rules. That is,* $\mathrm{vm}(\tilde{b}_1) = \mathrm{sr}(b_0)$ *and* $\mathrm{vm}(b_1) = \mathrm{sr}(\tilde{b}_0)$*.*

Vanishing Moments for Orthogonal Wavelet Filter Banks

Corollary

Let $\{b_0, b_1\}$ be an orthogonal wavelet filter bank. Then b_1 has m vanishing moments if and only if b_0 has m sum rules. That is, $\text{vm}(b_1) = \text{sr}(b_0)$.

We shall discuss multilevel discrete wavelet/framelet transform later and hence for a dual framelet filter bank $(\{\tilde{b}_0, \tilde{b}_1, \dots, \tilde{b}_s\}, \{b_0, b_1, \dots, b_s\})$, we define

$$\tilde{a} := \tilde{b}_0, \qquad \tilde{a} := b_0$$

for low-pass filters, because $\tilde{a}(1) \neq 0$ and $a(1) \neq 0$. We often normalize them so that $\tilde{a}(1) = a(1) = 1$, i.e., $\sum_{k\in\mathbb{Z}} \tilde{a}[k] = \sum_{k\in\mathbb{Z}} a[k] = 1$.
Hence, from now one, we shall use the notation

$$(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$$

for a dual framelet filter bank.

An Example of Tight Framelet Filter Banks

- A tight framelet filter bank $\{a; b_1, b_2\}$ is given by

$$a = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$

$$b_1 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, \tfrac{\sqrt{2}}{4}\}_{[-1,1]},$$

$$b_2 = \{-\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, -\tfrac{1}{4}\}_{[-1,1]}.$$

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- Note that $\mathsf{a}(z) = \frac{1}{4} z^{-1}(z+1)^2$, $\mathsf{b}_1(z) = \frac{\sqrt{2}}{2} z^{-1}(z-1)^2$, and $\mathsf{b}_2(\xi) = -\frac{1}{4} z^{-1}(z-1)^2$

$$\mathsf{a}(z)\mathsf{a}(z^{-1}) + \mathsf{b}_1(z)\mathsf{b}_1(z^{-1}) + \mathsf{b}_2(z)\mathsf{b}_2(z^{-1}) = 1,$$

$$\mathsf{a}(z)\mathsf{a}(-z^{-1}) + \mathsf{b}_1(z)\mathsf{b}_1(-z^{-1}) + \mathsf{b}_2(z)\mathsf{b}_2(-z^{-1}) = 0$$

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$$\mathrm{a}(z)\mathrm{a}(z^{-1}) + \mathrm{b}_1(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_2(z)\mathrm{b}_2(z^{-1}) = 1,$$

$$\mathrm{a}(z)\mathrm{a}(-z^{-1}) + \mathrm{b}_1(z)\mathrm{b}_1(-z^{-1}) + \mathrm{b}_2(z)\mathrm{b}_2(-z^{-1}) = 0$$

- $\mathrm{sr}(a) = 2$ and $\mathrm{vm}(b_1) = 1$, $\mathrm{vm}(b_2) = 2$.

An Example of Tight Framelet Filter Banks

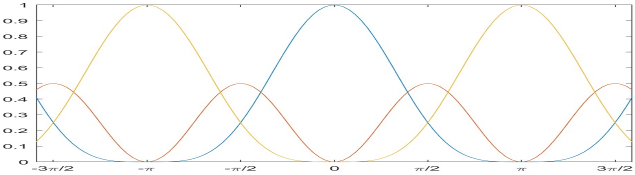
- A tight framelet filter bank $\{a; b_1, b_2\}$ is given by

$$a = \{\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]},$$
$$b_1 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, \tfrac{\sqrt{2}}{4}\}_{[-1,1]},$$
$$b_2 = \{-\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, -\tfrac{1}{4}\}_{[-1,1]}.$$

- Note that $\mathrm{a}(z) = \frac{1}{4}z^{-1}(z+1)^2$, $\mathrm{b}_1(z) = \frac{\sqrt{2}}{2}z^{-1}(z-1)^2$, and $\mathrm{b}_2(\xi) = -\frac{1}{4}z^{-1}(z-1)^2$

$$\mathrm{a}(z)\mathrm{a}(z^{-1}) + \mathrm{b}_1(z)\mathrm{b}_1(z^{-1}) + \mathrm{b}_2(z)\mathrm{b}_2(z^{-1}) = 1,$$
$$\mathrm{a}(z)\mathrm{a}(-z^{-1}) + \mathrm{b}_1(z)\mathrm{b}_1(-z^{-1}) + \mathrm{b}_2(z)\mathrm{b}_2(-z^{-1}) = 0$$

- $\mathrm{sr}(a) = 2$ and $\mathrm{vm}(b_1) = 1$, $\mathrm{vm}(b_2) = 2$.
- Frequency responses of $|\mathrm{a}(e^{-i\xi})|^2$ (blue), $|\mathrm{b}_1(e^{-i\xi})|^2$ (yellow) and $|\mathrm{b}_2(e^{-i\xi})|^2$ (red).



Discrete Framelet Transform using Tight Framelet Filter Banks

A test input data:

$$\mathring{v} = \{-21, -22, -23, -23, -25, 38, 36, 34\}_{[0,7]}$$

We extend $\mathring{v}$ to an 8-periodic sequence v on $\mathbb{Z}$, given by

$$v = \{\ldots, -25, 38, 36, 34, \underline{\mathbf{-21}, -22, -23, -23, -25, 38, 36, 34}, -21, -22, -23, -23, \ldots\}.$$

Then all sequences $\mathcal{T}_a v, \mathcal{T}_{b_1} v, \mathcal{T}_{b_2} v$ are 4-periodic and

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \underline{\mathbf{-15}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72,} -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \ldots\},$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \underline{\mathbf{-28}, -\tfrac{1}{2}, \tfrac{61}{2}, -2,} -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \ldots\},$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \underline{\mathbf{-27}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0,} -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \ldots\}.$$

It is also easy to check that $\frac{\sqrt{2}}{2}(\mathcal{S}_{b_0} w_0 + \mathcal{S}_{b_1} w_1 + \mathcal{S}_{b_2} w_2) = v$. But

$$\|w_0\|^2 + \|w_1\|^2 + \|w_2\|^2 = \frac{15571}{2} + \frac{3437}{2} + \frac{3571}{2} = \frac{22579}{4} \approx 5644.8$$

$$\|v\|^2 = 6504.$$

Example: LeGall Biorthogonal Wavelet Filter Bank

The LeGall biorthogonal wavelet filter bank is given by

$$a = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad \tilde{a} = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}$$

$$b = \{-\tfrac{1}{8}, \underline{\mathbf{-\tfrac{1}{4}}}, \tfrac{3}{4}, -\tfrac{1}{4}, -\tfrac{1}{8}\}_{[-1,3]}, \qquad \tilde{b} = \{\underline{\mathbf{-\tfrac{1}{4}}}, \tfrac{1}{2}, -\tfrac{1}{4}\}_{[0,2]},$$

Note that $\mathrm{sr}(a) = \mathrm{sr}(\tilde{a}) = 2$ and $\mathrm{vm}(b) = \mathrm{vm}(\tilde{b}) = 2$.
Extend $\mathring{v}$ by both endpoints non-repeated (EN):

$$v = \{\ldots, -25, -23, -23, -22, \underline{\mathbf{-21}, -22, -23, -23, -25, 38, 36, 34}, 36, 38, -25, -1, -1, \ldots$$

Then $\mathcal{T}_{\tilde{a}} v$ is 7-periodic and is symmetric about 0, 7/2:

$$w_0 = \frac{\sqrt{2}}{2}\mathcal{T}_{\tilde{a}} v = \frac{\sqrt{2}}{2}\{\ldots, -\frac{133}{4}, -\frac{91}{2}, \underline{\mathbf{-42}, -\frac{91}{2}, -\frac{133}{4}, \frac{349}{4}}, \frac{349}{4}, -\frac{133}{4}, \ldots\},$$

and $\mathcal{T}_{\tilde{b}} v$ is 7-periodic and is symmetric about $-\frac{1}{2}$, 3:

$$w_1 = \frac{\sqrt{2}}{2}\mathcal{T}_{\tilde{b}} v = \frac{\sqrt{2}}{2}\{\ldots, -2, \frac{65}{2}, 0, \underline{\mathbf{0}, 1, \frac{65}{2}, -2}, \frac{65}{2}, 1, 0, \ldots\}.$$

Example: Tight Framelet Filter Bank from B_2

Consider a tight framelet filter bank

$$a = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad b_1 = \{-\tfrac{\sqrt{2}}{4}, \underline{\mathbf{0}}, \tfrac{\sqrt{2}}{4}\}_{[-1,1]}, \quad b_2 = \{-\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, -\tfrac{1}{4}\}_{[-1,1]}.$$

Extend $\mathring{v}$ with both endpoints non-repeated (EN). Then all $\mathcal{T}_a v, \mathcal{T}_{b_1} v, \mathcal{T}_{b_2} v$ are 7-periodic and symmetric about 0 and $7/2$:

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, 72, -\tfrac{35}{2}, -\tfrac{91}{2}, \underline{\mathbf{-43}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72}, 72, -\tfrac{35}{2}, -\tfrac{91}{2}, \ldots\},$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, 2, -\tfrac{61}{2}, \tfrac{1}{2}, \underline{\mathbf{0}, -\tfrac{1}{2}, \tfrac{61}{2}, -2}, 2, -\tfrac{61}{2}, \tfrac{1}{2}, \ldots\}.$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, 0, -\tfrac{65}{2}, -\tfrac{1}{2}, \underline{\mathbf{1}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0}, 0, -\tfrac{65}{2}, -\tfrac{1}{2}, 1, \ldots\}.$$

Compare with framelet coefficients through periodic extension:

$$w_0 = \tfrac{\sqrt{2}}{2}\mathcal{T}_a v = \tfrac{\sqrt{2}}{2}\{\ldots, -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \underline{\mathbf{-15}, -\tfrac{91}{2}, -\tfrac{35}{2}, 72,} -15, -\tfrac{91}{2}, -\tfrac{35}{2}, 72, \ldots\},$$

$$w_1 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_1} v = \tfrac{\sqrt{2}}{2}\{\ldots, -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \underline{\mathbf{-28}, -\tfrac{1}{2}, \tfrac{61}{2}, -2,} -28, -\tfrac{1}{2}, \tfrac{61}{2}, -2, \ldots\},$$

$$w_2 = \tfrac{\sqrt{2}}{2}\mathcal{T}_{b_2} v = \{\ldots, -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \underline{\mathbf{-27}, -\tfrac{1}{2}, -\tfrac{65}{2}, 0,} -27, -\tfrac{1}{2}, -\tfrac{65}{2}, 0, \ldots\}.$$

Multi-level Fast Framelet Transform (FFrT)

- Let $\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}$ and $\{a; b_1, \dots, b_s\}$ be filters in $l_0(\mathbb{Z})$.
- For a positive integer J, a J-level discrete framelet decomposition is given by

$$v_j := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{a}} v_{j-1}, \qquad w_{\ell,j} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{b}_\ell} v_{j-1}, \quad \ell = 1, \dots, s, \; j = 1, \dots, J,$$

  where $v_0 : \mathbb{Z} \to \mathbb{C}$ is an input signal.
- $\tilde{\mathcal{W}}_J v_0 := (w_{1,1}, \dots, w_{s,1}, \dots, w_{1,J}, \dots, w_{s,J}, v_J)$.
- a J-level discrete framelet reconstruction is

$$v_{j-1} := \frac{\sqrt{2}}{2}\mathcal{S}_a v_j + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s} \mathcal{S}_{b_\ell} w_{\ell,j}, \quad j = J, \dots, 1.$$

- $\mathcal{V}_J(w_{1,1}, \dots, w_{s,1}, \dots, w_{1,J}, \dots, w_{s,J}, v_J) = v_0$.
- The perfect reconstruction property: $\mathcal{V}_J\tilde{\mathcal{W}}_J v_0 = v_0$ for all $J \in \mathbb{N}$, $v_0 \in l_2(\mathbb{Z})$.
- The fast framelet transform has the perfect reconstruction property if and only if $(\{\tilde{a}; \tilde{b}_1, \dots, \tilde{b}_s\}, \{a; b_1, \dots, b_s\})$ is a dual framelet filter bank satisfying

$$\begin{bmatrix} \hat{\tilde{a}}(z) & \hat{\tilde{b}}_1(z) & \cdots & \hat{\tilde{b}}_s(z) \\ \hat{\tilde{a}}(-z^{-1}) & \hat{\tilde{b}}_1(-z^{-1}) & \cdots & \hat{\tilde{b}}_s(-z^{-1}) \end{bmatrix} \begin{bmatrix} \hat{a}(z) & \hat{b}_1(z) & \cdots & \hat{b}_s(z) \\ \hat{a}(-z^{-1}) & \hat{b}_1(-z^{-1}) & \cdots & \hat{b}_s(-z^{-1}) \end{bmatrix}^{\star} = I_2.$$

- A fast framelet transform with $s = 1$ is called a fast wavelet transform.

Variants of FFrT: Undecimated FFrT

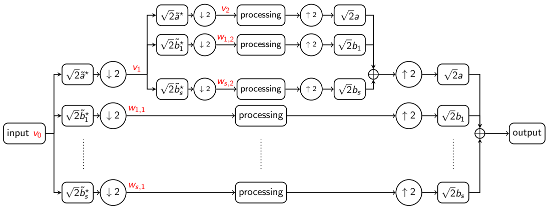
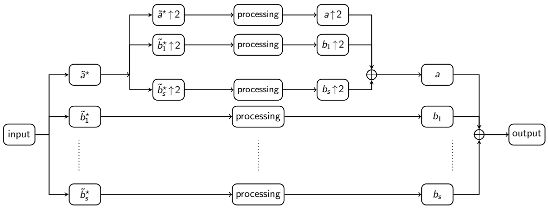


Figure: Diagram of a two-level discrete framelet transform using a pair of filter banks $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$.



Undecimated DFrT using a framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$, which is required to satisfy $\tilde{\mathrm{a}}(z)\mathrm{a}(z^{-1}) + \tilde{\mathrm{b}}_1(z)\mathrm{b}_1(z^{-1}) + \cdots + \tilde{\mathrm{b}}_s(z)\mathrm{b}_s(z^{-1}) = 1.$

Express J-level FFrT using Discrete Wavelets in $l_2(\mathbb{Z})$

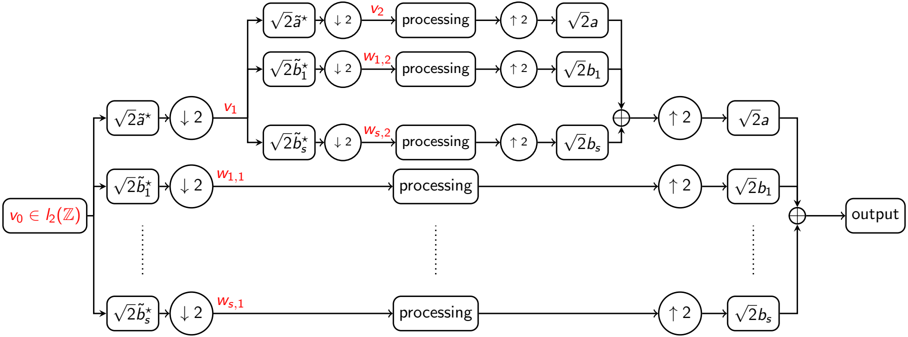


Figure: Diagram of a two-level discrete framelet transform using a pair of filter banks $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$.

Refinable Functions

- Let $a \in l_0(\mathbb{Z})$ with $\sum_{k\in\mathbb{Z}} a[k] = 1$.
- The refinable function $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi})$ is well defined for $\xi \in \mathbb{R}$ and satisfies

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} a[k]\phi(2x-k) \quad \text{i.e.,} \quad \widehat{\phi}(2\xi) = \mathrm{a}(e^{-i\xi})\widehat{\phi}(\xi).$$

Indeed,

$$\widehat{\phi}(2\xi) = \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{1-j}\xi}) = \mathrm{a}(e^{-i\xi}) \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi}) = \mathrm{a}(e^{-i\xi})\widehat{\phi}(\xi).$$

- Note that the Fourier transform of $\phi(2x-k)$ is

$$\phi\widehat{(2\cdot -k)}(\xi) = \int_{\mathbb{R}} \phi(2x-k)e^{-ix\xi}dx = \frac{1}{2}\int_{\mathbb{R}} \phi(y)e^{-i\frac{1}{2}(y+k)\xi}dy = \frac{1}{2}e^{-ik\xi}\widehat{\phi}(\xi/2).$$

Therefore, the Fourier transform of $2\sum_{k\in\mathbb{Z}} a[k]\phi(2x-k)$ is

$$2\sum_{k\in\mathbb{Z}} a[k]\frac{1}{2}e^{-ik\xi/2}\widehat{\phi}(\xi/2) = \sum_{k\in\mathbb{Z}} a[k]e^{-ik\xi/2}\widehat{\phi}(\xi/2) = \mathrm{a}(e^{-i\xi/2})\widehat{\phi}(\xi/2) = \widehat{\phi}(\xi).$$

This proves $2\sum_{k\in\mathbb{Z}} a[k]\phi(2x-k) = \phi(x)$.

Some Basics on Wavelets in $L_2(\mathbb{R})$

- For $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, define an affine system as

$$\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\}$$
$$\cup \{\psi^\ell_{2^j;k} := 2^{j/2}\psi^\ell(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}, \ell = 1, \ldots, s\}.$$

- We say that $\{\phi; \psi^1, \ldots, \psi^s\}$ is a framelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s)$ is a framelet in $L_2(\mathbb{R})$, that is, there exist positive constants $C_1, C_2 > 0$ such that

$$C_1\|f\|_2^2 \leqslant \sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{\ell=1}^{s}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{2^j;k}\rangle|^2 \leqslant C_2\|f\|_2^2, \quad \forall\, f \in L_2(\mathbb{R}).$$

- In particular, $\{\phi; \psi^1, \ldots, \psi^s\}$ is called a tight framelet in $L_2(\mathbb{R})$ if

$$\sum_{k\in\mathbb{Z}} |\langle f, \phi(\cdot - k)\rangle|^2 + \sum_{\ell=1}^{s}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}} |\langle f, \psi^\ell_{2^j;k}\rangle|^2 = \|f\|_2^2, \quad \forall\, f \in L_2(\mathbb{R}).$$

- Then $f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{2^j;k}\rangle \psi^\ell_{2^j;k}$.
- $\{\phi; \psi^1, \ldots, \psi^s\}$ is called an orthogonal wavelet in $L_2(\mathbb{R})$ if $\mathrm{AS}(\phi; \psi^1, \ldots, \psi^s)$ is an orthonormal basis in $L_2(\mathbb{R})$.
- $\{\phi; \psi^1, \ldots, \psi^s\}$ is an orthogonal wavelet in $L_2(\mathbb{R})$ if and only if it is a tight framelet in $L_2(\mathbb{R})$ and $\|\phi\|_2 = \|\psi_1\|_2 = \cdots = \|\psi^s\|_2 = 1$.

Dual Framelets in $L_2(\mathbb{R})$

For $\tilde{\phi}, \tilde{\psi}^1, \ldots, \tilde{\psi}^s \in L_2(\mathbb{R})$ and $\phi, \psi^1, \ldots, \psi^s \in L_2(\mathbb{R})$, we say that $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ is a dual framelet in $L_2(\mathbb{R})$ if

1. $\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}$ is a framelet in $L_2(\mathbb{R})$.
2. $\{\phi; \psi^1, \ldots, \psi^s\}$ is a framelet in $L_2(\mathbb{R})$.
3. The following identity holds:

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k)\rangle \langle \phi(\cdot - k), g\rangle + \sum_{\ell=1}^{s} \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle \langle \psi^{\ell}_{2^j;k}, g\rangle, \ \forall\, f, g \in L_2(\mathbb{R})$$

   with series converging absolutely.

Consequently, we have the wavelet representation of functions in $L_2(\mathbb{R})$:

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle \psi^{\ell}_{2^j;k}.$$

with the series converging unconditionally.

Characterization of Dual Framelets in $L_2(\mathbb{R})$

Theorem

Let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_s, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ *such that* $\mathrm{a}(1) = \tilde{\mathrm{a}}(1) = 1$. *Define* $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi})$, $\widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \tilde{\mathrm{a}}(e^{-i2^{-j}\xi})$ *and*

$$\widehat{\psi^\ell}(\xi) := \mathrm{b}_\ell(e^{-i\xi/2})\widehat{\phi}(\xi/2), \qquad \widehat{\tilde{\psi}^\ell}(\xi) := \tilde{\mathrm{b}}_\ell(e^{-i\xi/2})\widehat{\tilde{\phi}}(\xi/2), \qquad \ell = 1, \ldots, s.$$

Then the following are equivalent to each other

1. $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ *is a dual framelet in* $L_2(\mathbb{R})$.
2. $\phi, \tilde{\phi} \in L_2(\mathbb{R})$, $\mathrm{b}_1(1) = \cdots = \mathrm{b}_s(1) = 0$, $\tilde{\mathrm{b}}_1(1) = \cdots = \tilde{\mathrm{b}}_s(1) = 0$, *and* $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ *is a dual framelet filter bank, i.e.,*

$$\begin{bmatrix} \tilde{\mathrm{a}}(z) & \tilde{\mathrm{b}}_1(z) & \cdots & \tilde{\mathrm{b}}_s(z) \\ \tilde{\mathrm{a}}(-z^{-1}) & \tilde{\mathrm{b}}_1(-z^{-1}) & \cdots & \tilde{\mathrm{b}}_s(-z^{-1}) \end{bmatrix} \begin{bmatrix} \mathrm{a}(z) & \mathrm{b}_1(z) & \cdots & \mathrm{b}_s(z) \\ \mathrm{a}(-z^{-1}) & \mathrm{b}_1(-z^{-1}) & \cdots & \mathrm{b}_s(-z^{-1}) \end{bmatrix}^\star = I_2.$$

Wavelet Transform in $L_2(\mathbb{R})$

- Let $(\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}, \{\phi; \psi^1, \ldots, \psi^s\})$ is a dual framelet in $L_2(\mathbb{R})$ with a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$.
- For a given function $f \in L_2(\mathbb{R})$, we define

$$v^j(k) := \langle f, \tilde{\phi}_{2^j;k} \rangle, \qquad w^{\ell,j}(k) := \langle f, \tilde{\psi}^\ell_{2^j;k} \rangle, \qquad j, k \in \mathbb{Z}, \ell = 1, \ldots, s.$$

- They can be computed by fast wavelet transform:

$$v^{j-1} = \frac{\sqrt{2}}{2} \mathcal{T}_{\tilde{a}} v^j, \qquad w^{\ell,j-1} = \frac{\sqrt{2}}{2} \mathcal{T}_{\tilde{b}_\ell} v^j, \qquad \ell = 1, \ldots, s,$$

$$v^j = \frac{\sqrt{2}}{2} \mathcal{S}_a v^{j-1} + \sum_{\ell=1}^{s} \frac{\sqrt{2}}{2} \mathcal{S}_{\tilde{b}_\ell} w^{\ell,j-1}.$$

- For $J \in \mathbb{N}$, approximate $f \approx f_J := \sum_{k \in \mathbb{Z}} v_J(k) \phi_{2^J;k} = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}_{2^j;k} \rangle \phi_{2^j;k}$. Because $\int \tilde{\phi}(x) dx = \widehat{\tilde{\phi}}(0) = 1$, $\langle f, \tilde{\phi}_{2^j;k} \rangle \approx f(2^{-j}k) \langle 1, \tilde{\phi}_{2^j;k} \rangle = 2^{-J/2} f(2^{-j}k)$.
- $f_j = f_{j-1} + \sum_{\ell=1}^s \sum_{k \in \mathbb{Z}} w^{\ell,j-1} \psi^\ell_{2^{j-1};k} = f_{j-1} + \sum_{\ell=1}^s \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}^\ell_{2^{1-j};k} \rangle \psi^\ell_{2^{j-1};k}$.

$$f_J = f_0 + \sum_{\ell=1}^{s} \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} w^{\ell,j}(k) \psi_{2^j;k} = \sum_{k \in \mathbb{Z}} \langle f, \tilde{\phi}(\cdot - k) \rangle \phi(\cdot - k) + \sum_{\ell=1}^{s} \sum_{j=0}^{J-1} \sum_{k \in \mathbb{Z}} \langle f, \tilde{\psi}^\ell_{2^j;k} \psi_{2^j;k}.$$

Why Wavelets?

A wavelet ψ often has

1. compact support $\Rightarrow$ good spatial localization.
2. high smoothness/regularity $\Rightarrow$ good frequency localization.
3. high vanishing moments $\Rightarrow$ multiscale sparse representation. That is, most wavelet coefficients are small for smooth functions/signals.
4. associated filter banks $\Rightarrow$ fast wavelet transform to compute coefficients $\langle f, \psi^{\ell}_{2^j;k} \rangle$ through filter banks.
5. singularities of signals and their locations can be captured in large wavelet coefficients.
6. function spaces (Sobolev and Besov spaces) can be characterized by wavelets. This is important in harmonic analysis and numerical PDEs.

Explanation for Sparse Representation

- A wavelet function ψ has m vanishing moments if

$$\int_{\mathbb{R}} x^n \psi(x) dx = 0, \qquad n = 0, \ldots, m-1.$$

  That is, $\widehat{\psi}(0) = \widehat{\psi}'(0) = \cdots = \widehat{\psi}^{(m-1)}(0) = 0$. Define $\mathrm{vm}(\psi) := m$ largest.
- If $\widehat{\psi}(\xi) := \mathrm{b}(e^{-i\xi/2})\widehat{\phi}(\xi/2)$ and $\widehat{\phi}(0) \neq 0$, then $\mathrm{vm}(\psi) = \mathrm{vm}(b)$.
- The multiscale wavelet representation of $f \in L_2(\mathbb{R})$ is

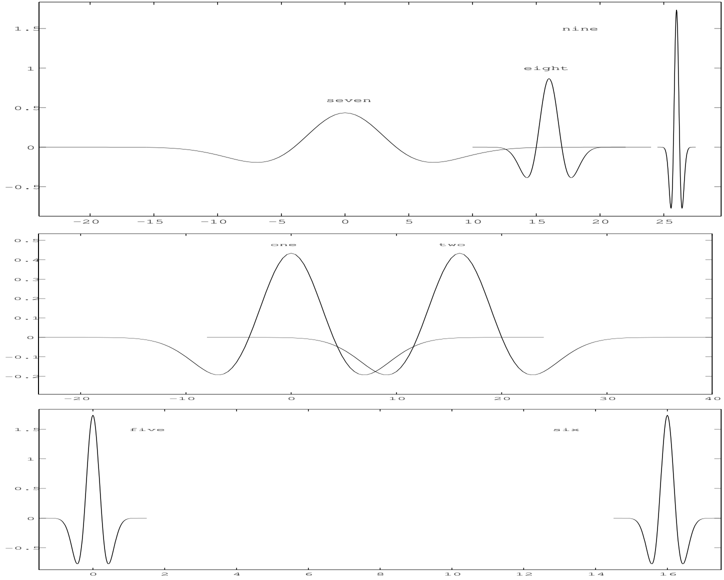
$$f = \sum_{k \in \mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \tilde{\phi}(\cdot - k) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \sum_{\ell=1}^{s} \langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle \psi^{\ell}_{2^j;k}$$

  with $\psi^{\ell}_{2^j;k}(x) := 2^{j/2}\psi^{\ell}(2^j x - k)$.
- $\mathrm{supp}\tilde{\psi}^{\ell}_{2^j;k} = 2^{-j}k + 2^{-j}\mathrm{supp}\tilde{\psi}^{\ell} \approx 2^{-j}k$ when $j \to \infty$.
- Wavelet coefficient $\langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle$ only depends f in the support of $\tilde{\psi}^{\ell}_{2^j;k}$. If f is smooth and can be well approximated by a polynomial P of degree $< m$, then

$$|\langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle| = |\langle f - P, \tilde{\psi}^{\ell}_{2^j;k}|\rangle = \|(f-P)\chi_{\mathrm{supp}(\tilde{\psi}^{\ell}_{2^j;k})}\|_2 \|\tilde{\psi}^{\ell}\|_2 \approx 0,$$

  where $\langle P, \psi^{\ell}_{2^j;k}\rangle = 2^{j/2}\int_{\mathbb{R}} P(x)\psi^{\ell}(2^j x - k)dx = 2^{-j/2}\int_{\mathbb{R}} P(2^{-j}(x+k))\psi^{\ell}(y)dy = 0$.
- If $\langle f, \tilde{\psi}^{\ell}_{2^j;k}\rangle$ is large for large j, we know the position of singularity, since $\mathrm{supp}\tilde{\psi}^{\ell}_{2^j\ k} = 2^{-j}\mathrm{supp}\tilde{\psi}^{\ell} + 2^{-j}k \approx 2^{-j}k$.

Dilates and Shifts of Multiscale Affine Systems



Tensor Product (Separable) Wavelets and Framelets in $\mathbb{R}^d$

- Let $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ be a dual framelet filter bank.
- Tensor product filters: $[u_1 \otimes \cdots \otimes u_d](k_1, \ldots, k_d) = u_1(k_1) \cdots u_d(k_d)$.
- Tensor product two-dimensional dual framelet filter bank:

$$\Big(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \otimes \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\}\Big).$$

  That is,

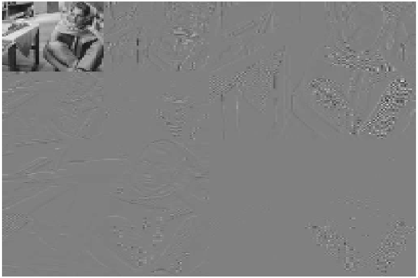
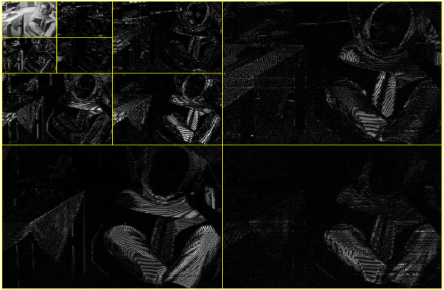
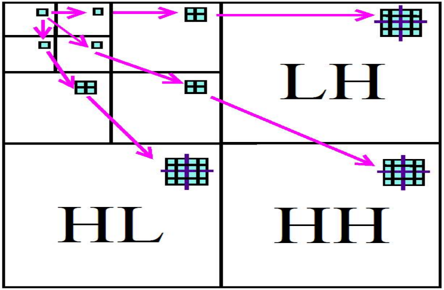
$$\{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\} = \{a \otimes a; b_1 \otimes a, \ldots, b_s \otimes a, \\ b_1 \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes\}$$

  consists of one low-pass tensor product filter $a \otimes a$ and total $(s+1)^2 - 1 = s^2 + 2s$ high-pass tensor product filters.
- Tensor product functions: $[f_1 \otimes \cdots \otimes f_d](x_1, \ldots, x_d) = f_1(x_1) \cdots f_d(x_d)$.
- Let $(\{\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_s\}, \{\phi; \psi_1, \ldots, \psi_s\})$ be a dual framelet in $L_2(\mathbb{R})$.
- Tensor product two-dimensional dual framelet in $L_2(\mathbb{R}^2)$:

$$\Big(\{\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_s\} \otimes \{\tilde{\phi}; \tilde{\psi}_1, \ldots, \tilde{\psi}_s\}, \{\phi; \psi_1, \ldots, \psi_s\} \otimes \{\phi; \psi_1, \ldots, \psi_s\}\Big).$$

- Advantages: fast and simple algorithm.

Sparsity and Multiscale Structure for Images

Connections of Tight Framelets and Tight Framelet Filter Banks

Theorem

Let $a, b_1, \dots, b_s \in l_0(\mathbb{Z})$ *with* $\sum_{k\in\mathbb{Z}} a[k] = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi}), \qquad \widehat{\psi^{\ell}}(\xi) := \mathrm{b}_{\ell}(e^{-i\xi/2})\widehat{\phi}(\xi/2), \qquad \ell = 1, \dots, s.$$

Then $\{\phi; \psi^1, \dots, \psi^s\}$ *is a tight framelet in* $L_2(\mathbb{R})$, *that is,*

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi(\cdot - k)\rangle \phi(\cdot - k) + \sum_{j=0}^{\infty} \sum_{\ell=1}^{s} \sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;k}\rangle \psi^{\ell}_{2^j;k}, \qquad \forall\, f \in L_2(\mathbb{R})$$

if and only if $\{a; b_1, \dots, b_s\}$ *is a tight framelet filter bank:*

$$\mathrm{a}(z)\mathrm{a}(-z^{-1}) + \sum_{\ell=1}^{s} \mathrm{b}_{\ell}(z)\mathrm{b}_{\ell}(-z^{-1}) = 1, \quad \mathrm{a}(z)\mathrm{a}(-z^{-1}) + \sum_{\ell=1}^{s} \mathrm{b}_{\ell}(z)\mathrm{b}_{\ell}(-z^{-1}) = 0.$$

The key: The tight framelet filter bank forces $\phi \in L_2(\mathbb{R})$ and $\mathrm{b}_1(1) = \cdots = \mathrm{b}_s(1) = 0$.

Orthogonal Wavelets vs Orthogonal Wavelet Filter Banks

Theorem

Let $a, b \in l_0(\mathbb{Z})$ *with* $\sum_{k\in\mathbb{Z}} a[k] = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi}), \qquad \widehat{\psi}(\xi) := \mathrm{b}(e^{-i\xi/2})\widehat{\phi}(\xi/2).$$

Then the following are equivalent to each other:

1. $\{\phi; \psi\}$ *is an orthogonal wavelet in* $L_2(\mathbb{R})$, *that is,*

   $$\mathrm{AS}(\phi;\psi) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot -k) \ : \ j \geqslant 0, k \in \mathbb{Z}\}$$

   is an orthonormal basis of $L_2(\mathbb{R})$.
2. $\{a; b\}$ *is an orthogonal wavelet filter bank and* $[\widehat{\phi}, \widehat{\phi}](\xi) = 1$ *almost everywhere (Note that* $[\widehat{\phi}, \widehat{\phi}] = 1 \iff \langle \phi, \phi(\cdot - k)\rangle = \boldsymbol{\delta}(k)$ *for* $k \in \mathbb{Z}$*)*
3. $\{a; b\}$ *is an orthogonal wavelet filter bank and* $\mathrm{sm}(a) > 0$.

For $f, g \in L_2(\mathbb{R})$, we define the bracket product to be

$$[f, g](\xi) := \sum_{k\in\mathbb{Z}} f(\xi + 2\pi k)\overline{g(\xi + 2\pi k)} = \left\langle \{f(\xi + 2\pi k)\}_{k\in\mathbb{Z}}, \{g(\xi + 2\pi k)\}_{k\in\mathbb{Z}} \right\rangle_{l_2(\mathbb{Z})}.$$

A Basic Identity

For $m, n \in \mathbb{N}$, $\mathrm{P}_{m,n}$ is the unique polynomial of degree at most $n-1$ satisfying

$$\mathrm{P}_{m,n}(x) := (1-x)^{-m} + \mathscr{O}(x^n), \quad x \to 0, \quad \text{that is,} \quad \mathrm{P}_{m,n}(x) = \sum_{j=0}^{n-1} \binom{m+j-1}{j} x^j.$$

Theorem

$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1$ *for all* $x \in \mathbb{R}, m \in \mathbb{N}$.

A Basic Identity

For $m, n \in \mathbb{N}$, $\mathrm{P}_{m,n}$ is the unique polynomial of degree at most $n-1$ satisfying

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Theorem

$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1$ *for all* $x \in \mathbb{R}, m \in \mathbb{N}$.

Proof. Define $\mathrm{P}(y,x) := \sum_{j=0}^{m-1} \binom{2m-1}{j} x^j y^{m-j-1}$. Then

$$(x+y)^{2m-1} = \sum_{j=0}^{2m-1} \binom{2m-1}{j} x^j y^{2m-1-j} = x^m \mathrm{P}(x,y) + y^m \mathrm{P}(y,x).$$

Note $\deg(\mathrm{P}(1-x,x)) < m$ and $x^m \mathrm{P}(x,1-x) + (1-x)^m \mathrm{P}(1-x,x) = 1$, from which we have

$$\begin{aligned} \mathrm{P}(1-x,x) &= (1-x)^{-m}[(1-x)^m \mathrm{P}(1-x,x)] = (1-x)^{-m}[1 - x^m \mathrm{P}(x,1-x)] \\ &= (1-x)^{-m} + \mathscr{O}(x^m), \quad x \to 0. \end{aligned}$$

By the uniqueness of $\mathrm{P}_{m,m}$, we must have $\mathrm{P}(x,1-x) = \mathrm{P}_{m,m}$. Hence, we proved

$$(1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1.$$

Construction of Interpolatory Filters

- A filter $a \in l_0(\mathbb{Z})$ is interpolatory if $\mathrm{a}(z) + \mathrm{a}(-z) = 1$, i.e.,

$$a[0] = \frac{1}{2} \quad \text{and} \quad a[2k] = 0, \quad \forall\, k \in \mathbb{Z}\backslash\{0\}.$$

- $\mathrm{a}(z) + \mathrm{a}(-z) = 2\sum_{k\in\mathbb{Z}} a[2k] z^{2k}$.
- $\frac{1}{4}z^{-1} + \frac{1}{2} + \frac{1}{4}z = 2^{-2}(1+z^{-1})(1+z)$.
- $2^{-2}(1+z^{-1})(1+z) = |(1+z)/2|^2 = \cos^2(\xi/2)$ for $z = e^{-i\xi}$.
- $-\frac{1}{4}z^{-1} + \frac{1}{2} - \frac{1}{4}z = 2^{-2}(1-z^{-1})(1-z)$.
- $2^{-2}(1-z^{-1})(1-z) = |(1-z)/2|^2 = \sin^2(\xi/2)$ for $z = e^{-i\xi}$.
- For $m \in \mathbb{N}$, a family of interpolatory filters a^I_{2m} is given by $\mathrm{a}^I_{2m}(e^{-i\xi}) :=$

$$\mathrm{a}^I_{2m}(e^{-i\xi}) := [2^{-2}(1+z^{-1})(1+z)]^m \mathrm{P}_{m,m}(2^{-2}(1-z^{-1})(1-z)).$$

$$\mathrm{a}^I_{2m}(e^{-i\xi}) := \cos^{2m}(\xi/2)\mathrm{P}_{m,m}(\sin^2(\xi/2)).$$

  Set $x = \sin^2(\xi/2)$. Then $\sin^2((\xi+\pi)/2) = \cos^2(\xi/2)$. Hence, for $z = e^{-i\xi}$,

$$\mathrm{a}^I_{2m}(z) = (1-x)^m \mathrm{P}_{m,m}(x) \quad \text{and} \quad \mathrm{a}^I_{2m}(-z) = x^m \mathrm{P}_{m,m}(1-x).$$

  Therefore, $\mathrm{a}^I_{2m}(z) + \mathrm{a}^I_{2m}(-z) = (1-x)^m \mathrm{P}_{m,m}(x) + x^m \mathrm{P}_{m,m}(1-x) = 1$.
- The mask a^I_{2m} has $2m$ sum rules satisfying $(1+z)^{2m} \mid \mathrm{a}^I_{2m}(z)$.
- Hence, $\mathrm{sr}(a^I_{2m}) = 2m$, $\mathrm{a}^I_{2m}(1) = 1$, and $\mathrm{fsupp}(a^I_{2m}) = [1-2m, 2m-1]$.
- $\mathrm{a}^I_{2m}(e^{-i\xi}) \geqslant 0$ for all $\xi \in \mathbb{R}$.
- The filters are called Deslauriers-Dubuc interpolatory filters.

Interpolatory Filters a^I_{2m}

$$a^I_2 = \{\tfrac{1}{4}, \underline{\tfrac{1}{2}}, \tfrac{1}{4}\}_{[-1,1]},$$

$$a^I_4 = \{-\tfrac{1}{32}, 0, \tfrac{9}{32}, \underline{\tfrac{1}{2}}, \tfrac{9}{32}, 0, -\tfrac{1}{32}\}_{[-3,3]},$$

$$a^I_6 = \{\tfrac{3}{512}, 0, -\tfrac{25}{512}, 0, \tfrac{75}{256}, \underline{\tfrac{1}{2}}, \tfrac{75}{256}, 0, -\tfrac{25}{512}, 0, \tfrac{3}{512}\}_{[-5,5]},$$

$$a^I_8 = \{-\tfrac{5}{4096}, 0, \tfrac{49}{4096}, 0, -\tfrac{245}{4096}, 0, \tfrac{1225}{4096}, \underline{\tfrac{1}{2}}, \tfrac{1225}{4096}, 0, -\tfrac{245}{4096}, 0, \tfrac{49}{4096}, 0, -\tfrac{5}{4096}\}_{[-7,7]}.$$

m	1	2	3	4	5
$\mathrm{sm}(a^I_{2m})$	1.5	2.440765	3.175132	3.793134	4.344084

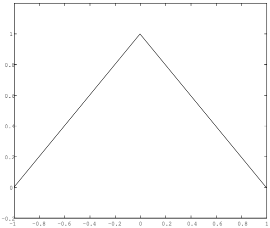
Theorem

Let $a \in l_0(\mathbb{Z})$ be interpolatory: $a[2k] = \frac{1}{2}\boldsymbol{\delta}[k]$ for $k \in \mathbb{Z}$. Define a refinable function by $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi})$ for $\xi \in \mathbb{R}$. If $\mathrm{sm}(a) > 1/2$, then ϕ is a compactly supported continuous function and is interpolating:

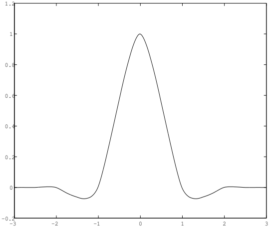
$$\phi(k) = \boldsymbol{\delta}[k], \qquad k \in \mathbb{Z}.$$

In particular, if $a = a^I_{2m}$ with $m \in \mathbb{N}$, then $\phi(k) = \boldsymbol{\delta}[k]$ for all $k \in \mathbb{Z}$.

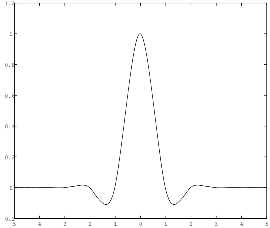
Compactly Supported Interpolating Function



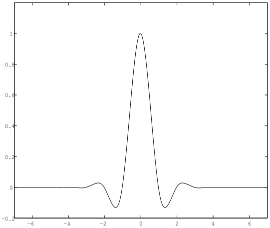
(a) $\phi^{a_2^l}$



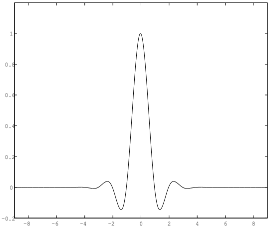
(b) $\phi^{a_4^l}$



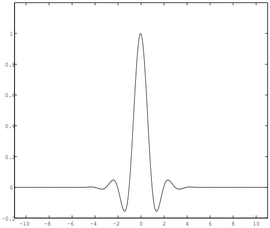
(c) $\phi^{a_6^l}$



(d) $\phi^{a_8^l}$



(e) $\phi^{a_{10}^l}$



(f) $\phi^{a_{12}^l}$

Fejér-Riesz Lemma

Lemma

Let $\boldsymbol{\Theta}$ be a 2π-periodic trigonometric polynomial with real coefficients (or with complex coefficients) such that $\boldsymbol{\Theta}(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$. Then there exists a 2π-periodic trigonometric polynomial $\boldsymbol{\theta}$ with real coefficients (or with complex coefficients) such that $|\boldsymbol{\theta}(\xi)|^2 = \boldsymbol{\Theta}(\xi)$ for all $\xi \in \mathbb{R}$. Moreover, if $\boldsymbol{\Theta}(0) \neq 0$, then we can further require $\boldsymbol{\theta}(0) = \sqrt{\boldsymbol{\Theta}(0)}$.

Definition: The Laurent polynomial of a filter $a = \{a[k]\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$ is $\mathsf{a}(z) := \sum_{k\in\mathbb{Z}} a[k]z^k$ for $z \in \mathbb{C}\backslash\{0\}$.

Fact: Let $\Theta(z)$ be the Laurent polynomial for $\boldsymbol{\Theta}$, i.e., $\boldsymbol{\Theta} = \Theta(e^{-i\xi})$. Define

$$Z := \{z \in \mathbb{C}\backslash\{0\} \; : \; \Theta(z) = 0\} \quad \text{counting multiplicity of zeros.}$$

Then $\boldsymbol{\Theta} \geqslant 0$ for all $\xi \in \mathbb{R} \iff Z$ is invariant under the mapping $z \mapsto \bar{z}^{-1}$ and any point in $Z \cap \mathbb{T}$ has even multiplicity. Then there is a unique subset $Y \subset Z \cap \{z \in \mathbb{C} \; : \; |z| \leqslant 1\}$ such that $Z = \{\zeta, \bar{\zeta}^{-1} \; : \; \zeta \in Y\}$. Define

$$\mathsf{p}(z) = \prod_{\zeta\in Y}(z - \zeta).$$

Then $\boldsymbol{\theta}(\xi) := ce^{-in\xi}\mathsf{p}(e^{-i\xi})$ satisfies $|\boldsymbol{\theta}(\xi)|^2 = \boldsymbol{\Theta}(\xi)$ for all $\xi \in \mathbb{R}$ with $c := \sqrt{\Theta(0)}/|\mathsf{p}(1)|$ and $n \in \mathbb{Z}$ (we often choose $n = 0$).

Daubechies Orthogonal Wavelets

Let a^I_{2m} be the interpolatory filter. Since $\mathrm{a}^I_{2m}(e^{-i\xi}) \geqslant 0$, by Fejér-Riesz lemma, there exists $a^D_m \in l_0(\mathbb{Z})$ such that $\mathrm{a}^D_m(1) = 1$ and

$$|\mathrm{a}^D_m(e^{-i\xi})|^2 = \mathrm{a}^I_{2m}(e^{-i\xi}) := \cos^{2m}(\xi/2)\mathrm{P}_{m,m}(\sin^2(\xi/2)).$$

Then $\mathrm{sr}(a^D_m) = m$ (i.e., a^D_m has m sum rules) and

$$|\mathrm{a}^D_m(z)|^2 + |\mathrm{a}^D_m(-z)|^2 = \mathrm{a}^I_{2m}(z) + \mathrm{a}^I_{2m}(-z) = 1.$$

Define ϕ through $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}^D_m(e^{-i2^{-j}\xi})$. Then we shall prove later that $\langle \phi(\cdot - k), \phi \rangle = \boldsymbol{\delta}[k]$ for all $k \in \mathbb{Z}$:

$$[\widehat{\phi}, \widehat{\phi}] := \sum_{k \in \mathbb{Z}} |\widehat{\phi}(\xi + 2\pi k)|^2 = 1$$

and $\{a^D_m; b^D_m\}$ is an orthogonal wavelet filter bank with

$$\mathrm{b}^D_m(z) := z\mathrm{a}^D_m(-z^{-1}).$$

Then $\mathrm{vm}(b^D_m) = m$ and $\{\phi; \psi\}$ is a compactly supported orthogonal wavelet, where

$$\widehat{\psi}(\xi) := \mathrm{b}^D_m(e^{-i\xi/2})\widehat{\phi}(\xi/2)$$

such that the low-pass filter a^D_m has order m sum rules and the high-pass filter b^D_m has m vanishing moments, called the Daubechies orthogonal wavelet of order m.

Daubechies Orthogonal Filters

$$a_1^D = \{\underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{2}\}_{[0,1]},$$

$$a_2^D = \{\tfrac{1+\sqrt{3}}{8}, \underline{\mathbf{\tfrac{3+\sqrt{3}}{8}}}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}_{[-1,2]}$$

$$a_3^D = \{\tfrac{1+\sqrt{10}+\sqrt{5+2\sqrt{10}}}{32}, \tfrac{5+\sqrt{10}+3\sqrt{5+2\sqrt{10}}}{32}, \underline{\mathbf{\tfrac{5-\sqrt{10}+\sqrt{5+2\sqrt{10}}}{16}}}, \tfrac{5-\sqrt{10}-\sqrt{5+2\sqrt{10}}}{16}, \tfrac{5+\sqrt{10}-3\sqrt{5+2\sqrt{10}}}{32}, \tfrac{1+\sqrt{10}-\sqrt{5+2\sqrt{10}}}{32}\}_{[-2,3]},$$

$$a_4^D = \{-0.0535744507091, -0.0209554825625, 0.351869534328, \underline{\mathbf{0.568329121704}}, 0.210617267102, -0.0701588120893, -0.00891235072084, 0.0227851729480\}_{[-3,4]}.$$

m	1	2	3	4	5	6
$\text{sm}(a_m^D)$	0.5	1.0	1.415037	1.775565	2.096787	2.388374

Example a_1^D Using the Interpolatory Filter a_2^I

- Consider $z := e^{-i\xi}$. Note $e^{-i(\xi+\pi)} = e^{-i\xi}e^{-i\pi} = -z$ and $e^{i\xi} = z^{-1}$.

$$\cos^2(\xi/2) = \frac{1}{2} + \frac{1}{4}z^{-1} + \frac{1}{4}z = 2^{-2}(1+z)(1+z^{-1}),$$

$$\sin^2(\xi/2) = \cos^2((\xi+\pi)/2) = \frac{1}{2} - \frac{1}{4}z^{-1} - \frac{1}{4}z = 2^{-2}(1-z)(1-z^{-1}).$$

- By definition, $\mathrm{a}_2^I(e^{-i\xi}) = \cos^2(\xi/2)\mathrm{P}_{1,1}(\sin^2(\xi/2))$ with $\mathrm{P}_{1,1}(x) = 1$.
- Hence, the Laurent polynomial representation of the filter a_2^I is

$$\mathrm{a}_2^I(z) = 2^{-2}(1+z)(1+z^{-1}).$$

Hence, The zero set Z of a_2^I is $\{-1,-1\}$ and we can take $Y = \{-1\}$. Define $\mathrm{p}(z) := z+1$ and $c := \sqrt{\widehat{a_2^I}(0)/|\mathrm{p}(1)|} = 1/2$.

- Hence, $a_1^D(z) = c\mathrm{p}(z) = 2^{-1}(1+z)$ and $\widehat{a_1^D}(\xi) = 2^{-1}(1+e^{-i\xi})$.
- $\mathrm{b}_1^D(z) := z\mathrm{a}_1^D(-z^{-1}) = 2^{-1}z(1-z^{-1}) = 2^{-1}(z-1)$.
- $a_1^D = \{\frac{1}{2}, \frac{1}{2}\}_{[0,1]}$ and $b_1^D = \{-\frac{1}{2}, \frac{1}{2}\}_{[0,1]}$ with $\mathrm{sr}(a_1^D) = 1$ and $\mathrm{vm}(b_1^D) = 1$.
- Fact: For the high-pass filter $\mathrm{b}(z) := z\mathrm{a}(-z^{-1})$, we have

$$\mathrm{b}(z) = z\mathrm{a}(-z^{-1}) = \sum_{k\in\mathbb{Z}} a(k)(-1)^k z^{1-k} = \sum_{n\in\mathbb{Z}} (-1)^{1-n} a(1-n) z^n.$$

Hence, $b(n) = (-1)^{1-n}a(1-n)$ for all $n \in \mathbb{Z}$.

Example a_2^D Using the Interpolatory Filter a_4^I

- By definition, $\mathrm{a}_4^I(e^{-i\xi}) = \cos^4(\xi/2)\mathrm{P}_{2,2}(\sin^2(\xi/2))$ with $\mathrm{P}_{2,2}(x) = 1 + 2x$.
- Hence, the Laurent polynomial representation of the filter a_4^I is

$$\mathrm{a}_4^I(z) = 2^{-4}(1+z)^2(1+z^{-1})^2\mathrm{q}(z) \quad \text{with} \quad \mathrm{q}(z) := 2 - \tfrac{1}{2}z - \tfrac{1}{2}z^{-1}.$$

- Because the root of $-2z\mathrm{q}(z) = z^2 - 4z + 1$ is $2 \pm \sqrt{3}$, the zero set Z of a_4^I is

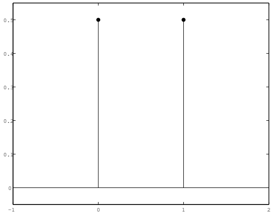
$$\{-1, -1, -1, -1, 2+\sqrt{3}, 2-\sqrt{3}\}$$

  and we can take $Y = \{-1, -1, 2-\sqrt{3}\}$. Define
  $\mathrm{p}(z) := (z+1)^2(z-2+\sqrt{3})$ and $c := \sqrt{\widehat{a_4^I}(0)}/|\mathrm{p}(1)| = \frac{\sqrt{3}+1}{8}$.
- Hence, we obtain the low-pass filter

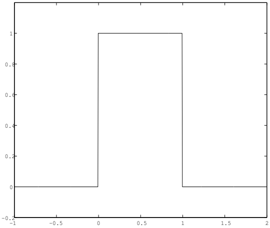
$$\begin{aligned}\mathrm{a}_2^D(z) = c\mathrm{p}(z) &= \frac{\sqrt{3}+1}{8}(1+z)^2(z-2+\sqrt{3}) \\ &= \frac{1-\sqrt{3}}{8} + \frac{3-\sqrt{3}}{8}z + \frac{3+\sqrt{3}}{8}z^3 + \frac{1+\sqrt{3}}{8}z^3.\end{aligned}$$

  Hence, $a_D^2 = \{\frac{1-\sqrt{3}}{8}, \frac{3-\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, \frac{1+\sqrt{3}}{8}\}_{[0,3]}$ and $\mathrm{sr}(a_2^D) = 2$.
- By $b(n) = (-1)^{1-n}a(1-n)$ for all $n \in \mathbb{Z}$, we have $\mathrm{vm}(b_2^D) = 2$ and $b_D^2 = \{-\frac{1+\sqrt{3}}{8}, \frac{3+\sqrt{3}}{8}, -\frac{3-\sqrt{3}}{8}, \frac{1-\sqrt{3}}{8}\}_{[-2,1]}$.
- We can also choose $Y = \{-1, -1, 2+\sqrt{3}\}$.

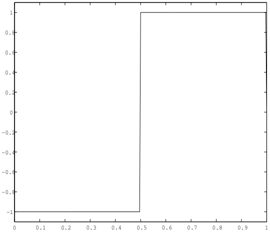
Daubechies Orthogonal Wavelets



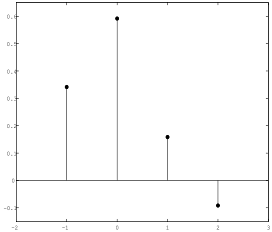
(a) Filter a_1^D



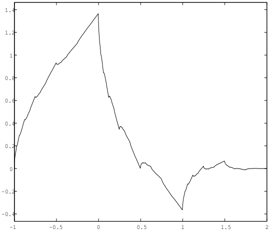
(b) $\phi^{a_1^D}$



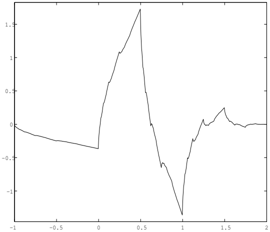
(c) $\psi^{a_1^D}$



(d) Filter a_2^D



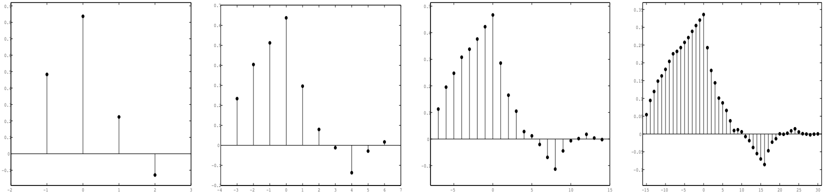
(e) $\phi^{a_2^D}$



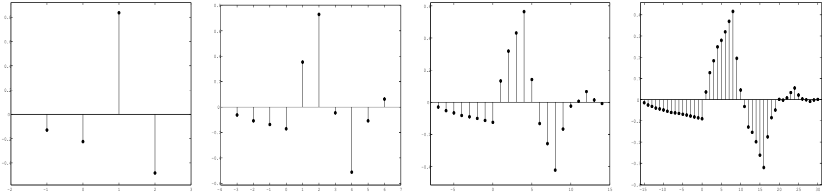
(f) $\psi^{a_2^D}$

An Example: Daubechies Orthogonal Wavelets

$$a = \{\tfrac{1+\sqrt{3}}{8}, \tfrac{3+\sqrt{3}}{8}, \tfrac{3-\sqrt{3}}{8}, \tfrac{1-\sqrt{3}}{8}\}, \quad b = \{-\tfrac{1-\sqrt{3}}{8}, \tfrac{3-\sqrt{3}}{8}, -\tfrac{3+\sqrt{3}}{8}, \tfrac{1+\sqrt{3}}{8}\}.$$



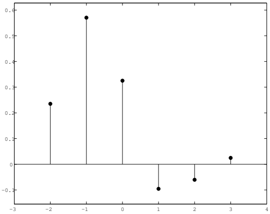
(g) a_1 (h) a_2 (i) a_3 (j) a_4



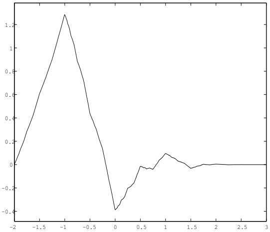
(k) b_1 (l) b_2 (m) b_3 (n) b_4

Figure: $\mathrm{DAS}_J(\{a; b\})$ is an orthonormal basis of $l_2(\mathbb{Z})$ for all $J \in \mathbb{N}$

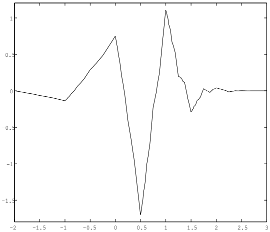
Daubechies Orthogonal Wavelets



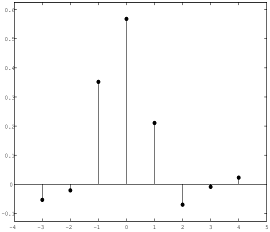
(a) Filter a_3^D



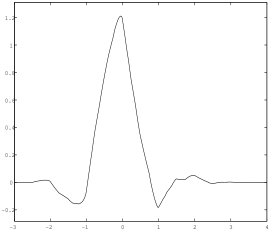
(b) $\phi^{a_3^D}$



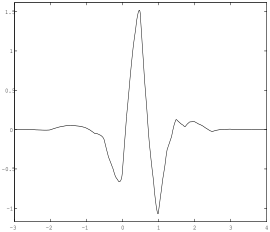
(c) $\psi^{a_3^D}$



(d) Filter a_4^D



(e) $\phi^{a_4^D}$



(f) $\psi^{a_4^D}$

Biorthogonal Wavelets in $L_2(\mathbb{R})$

- Let $\phi, \psi \in L_2(\mathbb{R})$ and $\tilde{\phi}, \tilde{\psi} \in L_2(\mathbb{R})$.
- $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ if
  1. Both $\{\tilde{\phi}; \tilde{\psi}\}$ and $\{\phi; \psi\}$ are Riesz wavelets in $L_2(\mathbb{R})$, i.e.,

$$C_1 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2 \leqslant \Big\| \sum_{h \in \mathrm{AS}(\phi;\psi)} c_h h \Big\|^2_{L_2(\mathbb{R})} \leqslant C_2 \sum_{h \in \mathrm{AS}(\phi;\psi)} |c_h|^2,$$

     where

$$\mathrm{AS}(\phi;\psi) := \{\phi(\cdot - k) \ : \ k \in \mathbb{Z}\} \cup \{\psi_{2^j;k} := 2^{j/2}\psi(2^j \cdot - k) \ : \ j \geqslant 0, k \in \mathbb{Z}\}.$$

  2. $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ and $\mathrm{AS}(\phi; \psi)$ are biorthogonal to each other:

$$\langle h, \tilde{h} \rangle = 1 \quad \text{and} \quad \langle h, g \rangle = 0, \quad \forall\, g \in \mathrm{AS}(\phi;\psi) \backslash \{h\}.$$

  3. The linear span of $\mathrm{AS}(\tilde{\phi}; \tilde{\psi})$ is dense in $L_2(\mathbb{R})$. The linear span of $\mathrm{AS}(\phi; \psi)$ is dense in $L_2(\mathbb{R})$.

Characterization of Biorthogonal Wavelets

Theorem

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$ *with* $\sum_{k\in\mathbb{Z}} a[k] = \sum_{k\in\mathbb{Z}} \tilde{a}[k] = 1$. *Define*

$$\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \mathrm{a}(e^{-i2^{-j}\xi}), \qquad \widehat{\psi}(\xi) := \mathrm{b}(e^{-i\xi/2})\widehat{\phi}(\xi/2),$$

$$\widehat{\tilde{\phi}}(\xi) := \prod_{j=1}^{\infty} \tilde{\mathrm{a}}(e^{-i2^{-j}\xi}), \qquad \widehat{\tilde{\psi}}(\xi) := \tilde{\mathrm{b}}(e^{-i\xi/2})\widehat{\tilde{\phi}}(\xi/2).$$

Then $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ *is a biorthogonal wavelet in* $L_2(\mathbb{R})$ *if and only if* $\mathrm{sm}_2(a) > 0$, $\mathrm{sm}_2(\tilde{a}) > 0$, *and* $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ *is a biorthogonal wavelet filter bank:*

$$\begin{bmatrix} \tilde{\mathrm{a}}(z) & \tilde{\mathrm{b}}(z) \\ \tilde{\mathrm{a}}(-z) & \tilde{\mathrm{b}}(-z) \end{bmatrix} \begin{bmatrix} \mathrm{a}(z^{-1}) & \mathrm{b}(z^{-1}) \\ \mathrm{a}(-z^{-1}) & \mathrm{b}(-z^{-1}) \end{bmatrix}^{\mathsf{T}} = I_2.$$

Construction of Biorthogonal Wavelet Filter Bank

Proposition

Let $a, b, \tilde{a}, \tilde{b} \in l_0(\mathbb{Z})$. Then $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank:

$$\begin{bmatrix} \tilde{a}(z) & \tilde{b}(z) \\ \tilde{a}(-z) & \tilde{b}(-z) \end{bmatrix} \begin{bmatrix} a(z^{-1}) & b(z^{-1}) \\ a(-z^{-1}) & b(-z^{-1}) \end{bmatrix}^{T} = I_2$$

if and only if $(\tilde{a}, a)$ is a biorthogonal low-pass filter:

$$\tilde{a}(z)a(z^{-1}) + \tilde{a}(-z)a(-z^{-1}) = 1$$

and there exist $c \neq 0$ and $n, \tilde{n} \in \mathbb{Z}$ such that

$$\tilde{b}(z) = cz^{1-2n}a(-z^{-1}), \qquad b(z) = c^{-1}z^{-(2\tilde{n}-1)}\tilde{a}(-z^{-1}).$$

- We often take

$$\tilde{b}(z) = za(-z^{-1}), \qquad b(z) = z\tilde{a}(-z^{-1}).$$

- $(\tilde{a}, a)$ is a biorthogonal low-pass filter if and only if $c := a^{\star} * \tilde{a}$ is an interpolatory mask.

Example of Biorthogonal Wavelets

We can obtain a pair of biorthogonal wavelet filters by splitting interplatory filters

$$\tilde{a}_m(z^{-1})a_m(z) := a^I_{2m}(z) = \cos^{2m}(\xi/2)P_{m,m}(\sin^2(\xi/2))$$

as follows: $P(x)\tilde{P}(x) = P_{m,m}(x)$ and

$$a_m(z) = 2^{-m}z^{-\lfloor m/2\rfloor}(1+z)^m P(\sin^2(\xi/2)), \qquad b_m(z) := z\tilde{a}_m(-z^{-1}),$$
$$\tilde{a}_m(z) = 2^{-m}z^{-\lfloor m/2\rfloor}(1+z)^m \tilde{P}(\sin^2(\xi/2)), \qquad \tilde{b}_m(z) := z a_m(-z^{-1}).$$

The functions in a biorthogonal wavelet $(\{\tilde{\phi};\tilde{\psi}\},\{\phi;\psi\})$ are defined by

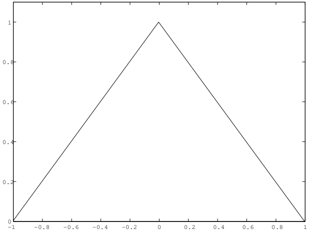
$$\widehat{\phi}(\xi) = \prod_{j=1}^{\infty} a_m(e^{-i2^{-j}\xi}), \qquad \widehat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \tilde{a}_m(e^{-i2^{-j}\xi}),$$
$$\widehat{\psi}(\xi) = b_m(e^{-i\xi/2})\widehat{\phi}(\xi/2), \quad \widehat{\tilde{\psi}}(\xi) = \tilde{b}_m(e^{-i\xi/2})\widehat{\tilde{\phi}}(\xi/2).$$

For $m=1$, we have $P_{1,1}(x)=1$. Taking $P(x)=1$ and $\tilde{P}(x)=1$, we have the Haar orthogonal wavelet filter bank.
For $m=2$, we have $P_{2,2}(x)=1+2x$. Taking $P(x)=1$ and $\tilde{P}(x)=1+2x$, we have the LeGall biorthogonal wavelet filter bank:

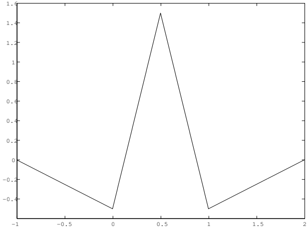
$$a_2 = \{\tfrac{1}{4}, \underline{\mathbf{\tfrac{1}{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad \tilde{a}_2 = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{3}{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}.$$

Note that $\mathrm{sm}(a_2) = 1.5$ and $\mathrm{sm}(\tilde{a}_2) = 0.440765$.

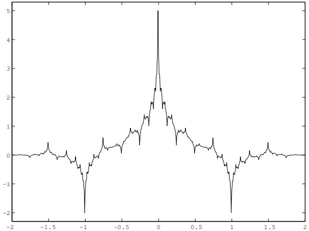
Examples: LeGall Biorthogonal Wavelet



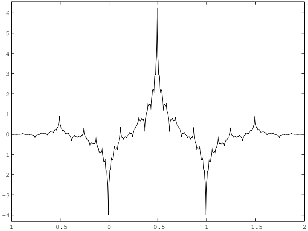
(g) ϕ



(h) ψ



(i) $\tilde{\phi}$



(j) $\tilde{\psi}$

The Most Famous Biorthogonal Wavelet

For $m = 4$, we have

$$\mathsf{P}_{4,4}(x) = 1 + 4x + 10x^2 + 20x^3.$$

Picking $\mathsf{P}(x) = 1 + tx$ and $\tilde{\mathsf{P}}(x) = 1 + (4-t)x + (10 - 4t + t^2)x^2$ with $t := \frac{30t_0}{t_0^2 + 5t_0 - 35} \approx 2.92069$ with $t_0 = (350 + 105\sqrt{15})^{1/3}$, we have

$$a_4 = \left\{-\frac{t}{64}, \frac{2-t}{32}, \frac{16+t}{64}, \underline{\mathbf{\frac{6+t}{16}}}, \frac{16+t}{64}, \frac{2-t}{32}, -\frac{t}{64}\right\}_{[-3,3]},$$

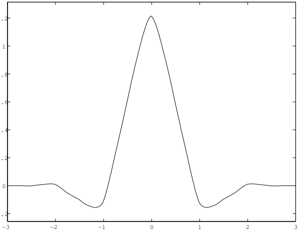
$$\tilde{a}_4 = \left\{\frac{t^2-4t+10}{256}, \frac{t-4}{64}, \frac{-t^2+6t-14}{64}, \frac{20-t}{64}, \underline{\mathbf{\frac{3t^2-20t+110}{128}}}, \frac{20-t}{64}, \frac{-t^2+6t-14}{64}, \frac{t-4}{64}, \frac{t^2-4t+10}{256}\right\}_{[-4,4]}.$$

The derived biorthogonal wavelet is called Daubechies 7/9 filter and has very impressive performance in many applications.

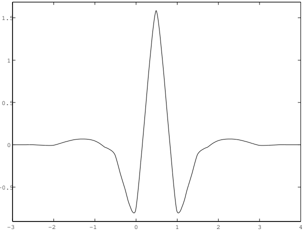
Note that

$$\mathrm{sm}(a) \approx 2.122644, \qquad \mathrm{sm}(\tilde{a}) \approx 1.409968.$$

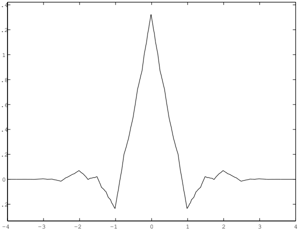
Example: Daubechies 7/9 Biorthogonal Wavelets



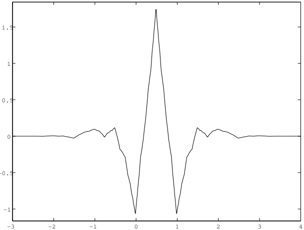
(k) ϕ



(l) ψ



(m) $\tilde{\phi}$



(n) $\tilde{\psi}$

Example from a_2^B

Let

$$a_2^B = \{\underline{\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]}$$

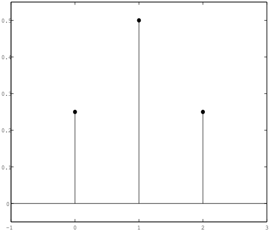
be the B-spline filter of order 2. Let

$$b_1 = \{\underline{-\mathbf{\tfrac{\sqrt{2}}{4}}}, 0, \tfrac{\sqrt{2}}{4}\}_{[0,2]},$$

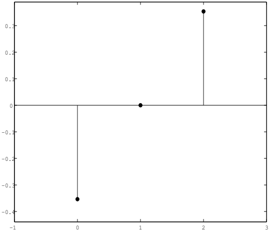
$$b_2 = \{\underline{-\mathbf{\tfrac{1}{4}}}, \tfrac{1}{2}, -\tfrac{1}{4}\}_{[0,2]}.$$

Then $\{a_2^B; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 2 sum rules and both b_1, b_2 have 1 vanishing moments.

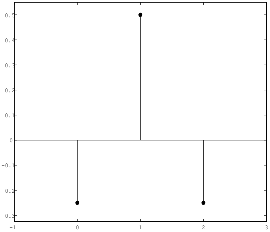
Tight Framelet from B_2



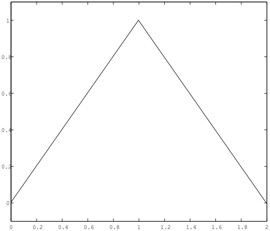
(a) Filter a_2^B



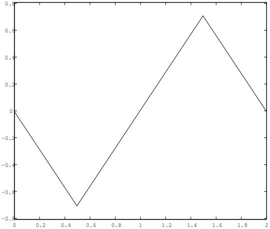
(b) Filter b_1



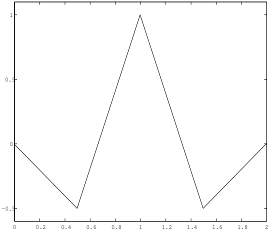
(c) Filter b_2



(d) B_2



(e) ψ^1



(f) ψ^2

Example from B_3

Let

$$a_3^B = \{\underline{\mathbf{\tfrac{1}{8}}}, \tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

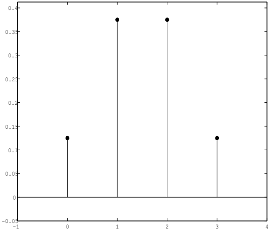
be the B-spline filter of order 3. Let

$$b_1 = \tfrac{\sqrt{3}}{4}\{\underline{-\mathbf{1}}, 1\}_{[0,1]},$$

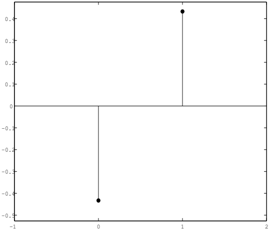
$$b_2 = \{\underline{-\mathbf{\tfrac{1}{8}}}, -\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[0,3]}$$

Then $\{a; b_1, b_2\}$ is a tight framelet filter bank such that a_2^B has order 3 sum rules and both b_1, b_2 have 1 vanishing moments.

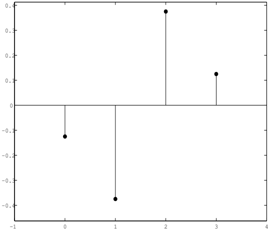
Tight Framelet from B_3



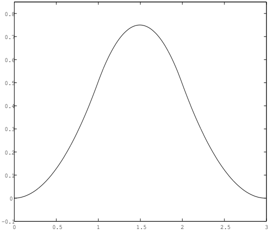
(a) Filter a_3^B



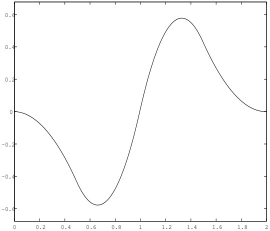
(b) Filter b_1



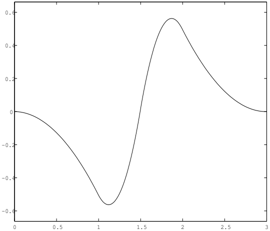
(c) Filter b_2



(d) B_3



(e) ψ^1



(f) ψ^2

Wavelet applications

The general procedure of wavelet applications in signal and image processing: For an input data v,

- Perform multi-level wavelet/framelet decomposition: $w = \mathcal{W}v$.
- Process the wavelet coefficients w to obtain new wavelet coefficients $\mathring{w}$.
- Perform multi-level wavelet/framelet reconstruction $\mathring{v} = \mathcal{V}\mathring{w}$.

Most data are supported on a bounded interval and we have to perform fast wavelet/framelet transform (FFrT) on data on bounded intervals!.

Variants of FFrT: Undecimated FFrT

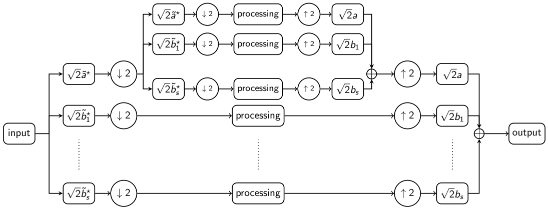
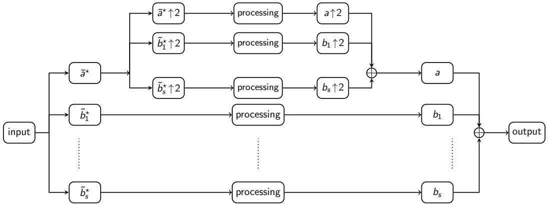


Figure: Diagram of a two-level discrete framelet transform using a pair of filter banks $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$.



Undecimated DFrT using a framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, (a; b_1, \ldots, b_s\})$, which is required to satisfy $\tilde{\mathrm{a}}(z)\mathrm{a}(z^{-1}) + \tilde{\mathrm{b}}_1(z)\mathrm{b}_1(z^{-1}) + \cdots + \tilde{\mathrm{b}}_s(z)\mathrm{b}_s(z^{-1}) = 1.$

Processing Wavelet Coefficients

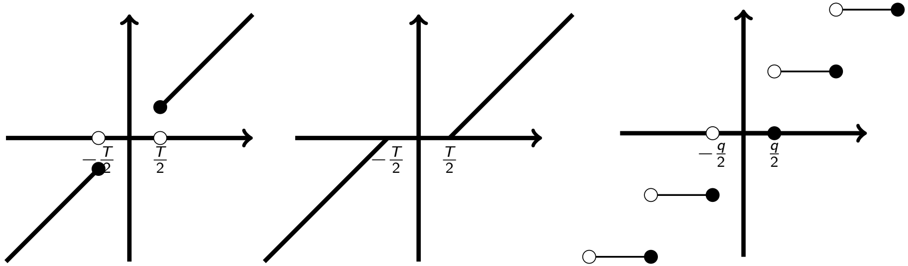


Figure: The hard thresholding, soft thresholding, and quantization.

Quantization is used for compression: convert a real number into a discrete set $\{\ldots, -2q, -q, 0, q, 2q, \ldots\}$.
The hard or soft thresholding is used for denoising and processing.

Tensor Product (Separable) Tight Framelet

- Let $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})$ be a 1D dual framelet filter bank.
- If $s = 1$, it is called a biorthogonal wavelet filter bank.
- Tensor product filters: $[u_1 \otimes \cdots \otimes u_d](k_1, \ldots, k_d) = u_1(k_1) \cdots u_d(k_d)$.
- Tensor product two-dimensional dual framelet filter bank:

$$\Big(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\} \otimes \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\}\Big).$$

  That is,

$$\{a; b_1, \ldots, b_s\} \otimes \{a; b_1, \ldots, b_s\} = \{a \otimes a; b_1 \otimes a, \ldots, b_s \otimes a, \\ b_1 \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes b_1, \ldots, b_s \otimes\}$$

  consists of one low-pass tensor product filter $a \otimes a$ and total $(s+1)^2 - 1 = s^2 + 2s$ high-pass tensor product filters.
- Advantages: fast and simple algorithm.

Wavelets for Image/Signal Compression

- The most popular wavelets used for the compression purpose are biorthogonal wavelet filter banks with symmetry and $4 \sim 6$ vanishing moments, in particular, LeGall and Daubechies 7/9 biorthogonal wavelet filter banks.
- For filter banks having symmetry, signals on intervals and images on rectangles are often extended by symmetry extension.
- Dauchies orthogonal wavelet filter banks are also used. Due to lack of symmetry, periodic extension of data on intervals and rectangles is often used.
- Grey scale images I : each entry of I takes discrete values $[0, 1, \ldots, 255]$ ($2^8 = 256$).
- Color images I has three channel: Red (R), Green (G), Blue (B). With each entry in each channel takes values discrete values $[0, 1, \ldots, 255]$.

Most Popular Wavelets for Compression

- LeGall biorthogonal wavelet filter bank $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$:

$$a = \{\tfrac{1}{4}, \underline{\tfrac{\mathbf{1}}{\mathbf{2}}}, \tfrac{1}{4}\}_{[-1,1]}, \qquad \tilde{a} = \{-\tfrac{1}{8}, \tfrac{1}{4}, \underline{\tfrac{\mathbf{3}}{\mathbf{4}}}, \tfrac{1}{4}, -\tfrac{1}{8}\}_{[-2,2]}$$

$$b = \{\tfrac{1}{8}, \underline{\tfrac{\mathbf{1}}{\mathbf{4}}}, -\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{8}\}_{[-1,3]}, \quad \tilde{b} = \{\underline{\tfrac{\mathbf{1}}{\mathbf{4}}}, -\tfrac{1}{2}, \tfrac{1}{4}\}_{[0,2]},$$

  where $b[k] = (-1)^{1-k}\overline{\tilde{a}[1-k]}$ and $\tilde{b}[k] = (-1)^{1-k}a[1-k]$. Both a and $\tilde{a}$ have order 2 sum rules, while both b and $\tilde{b}$ have 2 vanishing moments. Use $\{a; b\}$ for reconstruction.

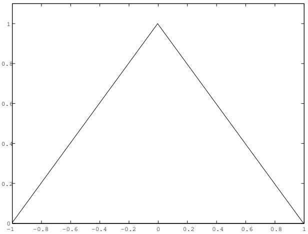
- Dauchies 7/9 biorthogonal wavelet filter bank $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$:

$$a = \{-\tfrac{t}{64}, \tfrac{2-t}{32}, \tfrac{16+t}{64}, \underline{\tfrac{\mathbf{6+t}}{\mathbf{16}}}, \tfrac{16+t}{64}, \tfrac{2-t}{32}, -\tfrac{t}{64}\}_{[-3,3]},$$

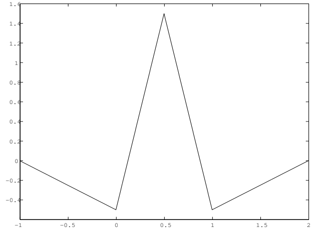
$$\tilde{a} = \{\tfrac{t^2-4t+10}{256}, \tfrac{t-4}{64}, \tfrac{-t^2+6t-14}{64}, \tfrac{20-t}{64}, \underline{\tfrac{\mathbf{3t^2-20t+110}}{\mathbf{128}}}, \tfrac{20-t}{64}, \tfrac{-t^2+6t-14}{64}, \tfrac{t-4}{64}, \tfrac{t^2-4t+10}{256}\}_{[-4,4]},$$

  where $t \approx 2.92069$ such that both a and $\tilde{a}$ have order 4 sum rules, while both b and $\tilde{b}$ have 4 vanishing moments.

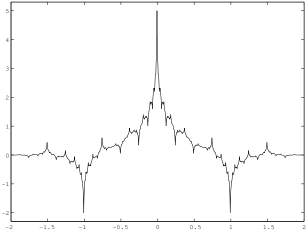
LeGall Biorthogonal Wavelets



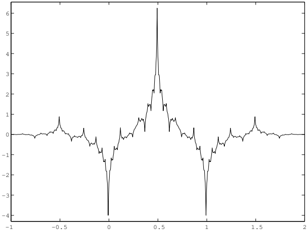
(a) ϕ



(b) ψ

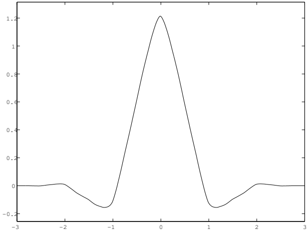


(c) $\tilde{\phi}$

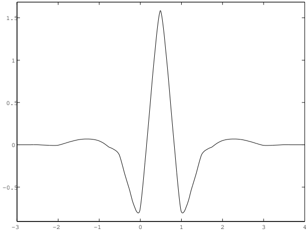


(d) $\tilde{\psi}$

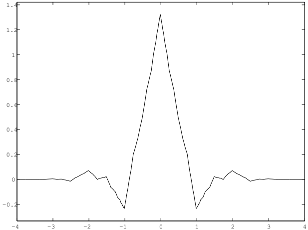
Daubechies 7/9 Biorthogonal Wavelets



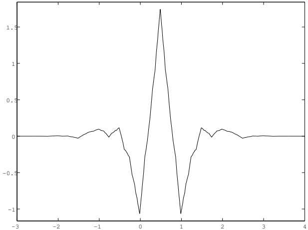
(e) ϕ



(f) ψ



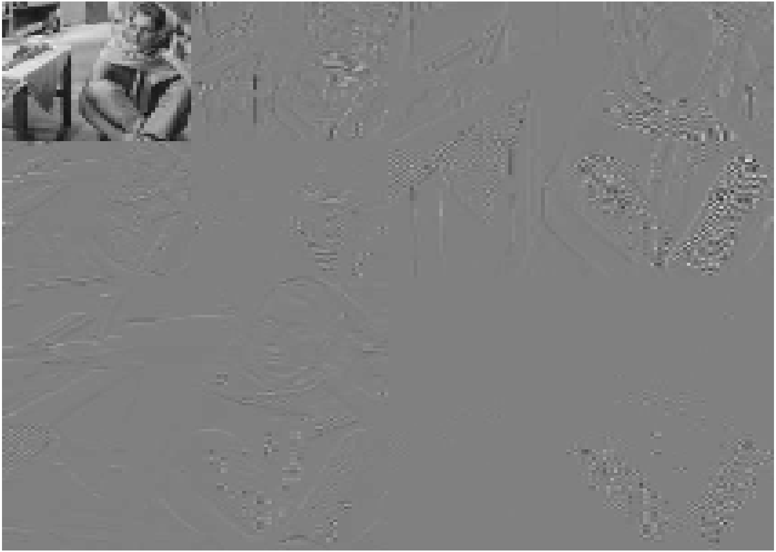
(g) $\tilde{\phi}$



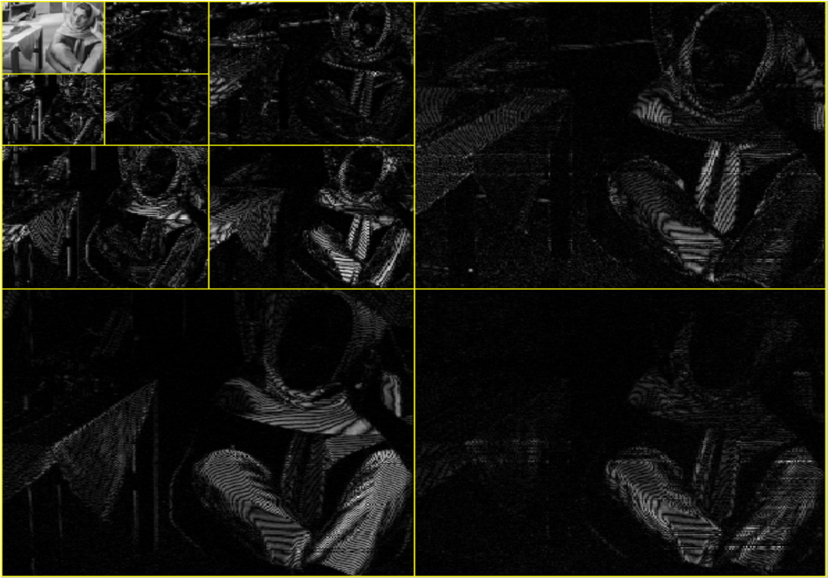
(h) $\tilde{\psi}$

Test Image: Barbara

Tree Structure of Wavelet coefficients



Tree Structure of Wavelet coefficients



Tree structure of wavelet coefficients

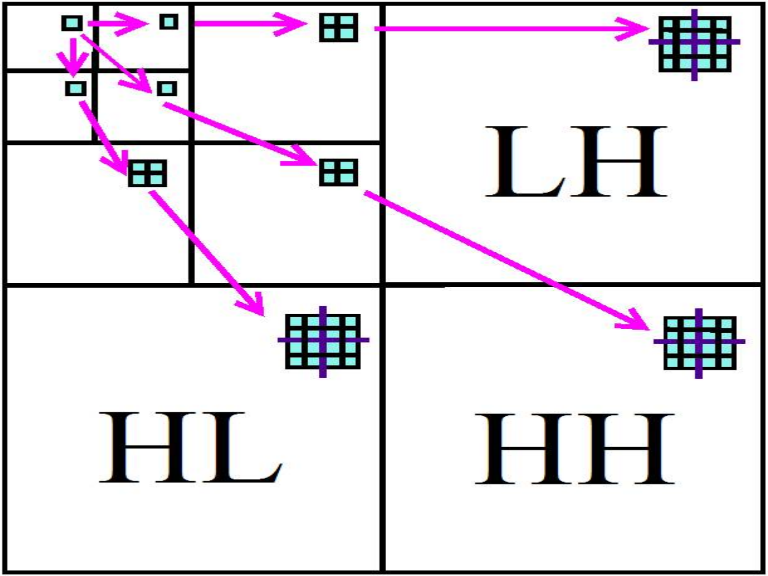


Image compression by wavelets

The original image of Lena

Image compression by wavelets

Compressed Lena image with compression ratio 32

Image compression by wavelets

Compressed Lena image with compression ratio 128

For Other Signal and Image Processing

- Beyond compression purpose, framelet filter banks are often used instead.
- Orthogonal wavelets and biorthogonal wavelets suffer a few key desired properties: translation invariant (see next page from Selesnick's paper) and directionality.
- For image denoising, people often use undecimated wavelet transforms, which are just special cases of framelets.
- Processing wavelet or framelet coefficients through thresholding is a key issue. Statistics and probability theory are often involved.

Signal/Image/Video Denoising and Inpainting

- Let $\mathrm{g} = (\mathrm{g}_1, \ldots, \mathrm{g}_d)^{\mathsf{T}}$ be an observed corrupted data:

$$\mathrm{g}_j = \begin{cases} \mathrm{f}_j + \mathrm{n}_j, & \text{if } j \in \Omega, \\ \mathrm{m}_j, & \text{if } j \in \Omega^c := \{1, \ldots, d\} \backslash \Omega, \end{cases}$$

- f=true data (signal/image.video).
- $\Omega \subseteq \{1, \ldots, d\}$ is a (known or unknown) observable region.
- n= i.i.d. Gaussian noise with zero mean and variance σ^2.
- m= either unknown missing pixel or impulse noise.
- Goal: Recover the true unknown data f from the corrupted g by suppressing noise n or inpainting unknown m.
- Denoising problem if $\Omega = \{1, \ldots, d\}$.
- Inpainting problem if Ω is known.
- Removing mixed noise problem if Ω is unknown.

Image Denoising

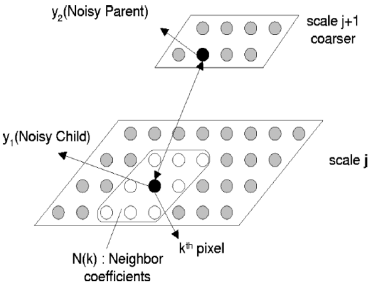
- Image denoising model is

$$\mathbf{z} = \mathbf{x} + \mathbf{n}.$$

  where
- $\mathbf{z}$=noisy image,
- $\mathbf{x}$=true image,
- $\mathbf{n}$=i.i.d. Gaussian noise of zero mean and variance σ^2.
- Coefficients $\mathbf{y} = D^{\mathsf{T}}\mathbf{z}$ after transform with a tight frame D.
- Perform thresholding on $\mathbf{y}$ to get $\tilde{\mathbf{y}}$.
- Take inverse transform to get a reconstructed image $\tilde{\mathbf{x}} = D\tilde{\mathbf{y}}$.

Bivariate Shrinkage

- Image model $\mathbf{z} = \mathbf{x} + \mathbf{n}$. where $\mathbf{z}$=noisy image, $\mathbf{x}$=true image, and $\mathbf{n}$=i.i.d. Gaussian noise of zero mean and variance σ^2.
- Coefficients $\mathbf{y} = D\mathbf{z}$ after transform with transform/dictionary D
- Bivariate shrinkage by Selesnick: $T = \frac{\sqrt{3}\sigma^2}{\sigma_x\sqrt{y_1^2+y_2^2}}$.



This figure is from Selesnick's paper.

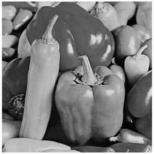
Test Images, Videos, and Inpainting Masks

(i) Barbara

(j) Lena

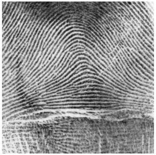
(k) C.man

(l) House



(m) Peppers

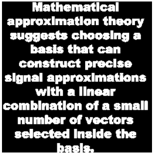
(n) Boat



(o) F.print

(p) Mobile

(q) C.guard



(r) Mask

Denoising Results: Barbara

	Barbara					
σ	UDWT	DT	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)	$\mathbb{C}$TF4	$\mathbb{C}$TF6(Gain)
5	36.90	37.36	37.41	**37.82**(0.46)	37.75	**38.10**(0.74)
10	32.66	33.52	33.62	**34.14**(0.48)	34.10	**34.47**(0.95)
15	30.31	31.38	31.47	**32.02**(0.64)	31.97	**32.32**(0.94)
20	28.71	29.87	29.91	**30.49**(0.62)	30.43	**30.77**(0.90)
25	27.52	28.70	28.71	**29.31**(0.61)	29.26	**29.57**(0.87)
30	26.58	27.77	27.74	**28.34**(0.57)	28.32	**28.61**(0.84)
50	24.27	25.26	25.21	**25.71**(0.45)	25.69	**26.02**(0.76)

The larger PSNR ($= 10\log_{10}\frac{255^2}{MSE}$) the better performance, where $MSE(u,v) := \frac{1}{|S|}\sum_{k\in S}|u(k)-v(k)|^2$ is the mean squared error.

Denoising Results: Barbara

The original image is on the left, the noisy image with $\sigma = 30$ is in the middle, and the denoised image is on the right.

Image Inpainting Model: $\mathbf{y} = \chi_\Omega \mathbf{x} + \mathbf{n}$

- Let $\Omega \subseteq \{1, \ldots, d\}$ be an observable region.

$$y_j = \begin{cases} x_j + n_j, & j \in \Omega, \\ \text{arbitray (unknown)}, & j \notin \Omega. \end{cases}$$

- $\mathbf{y} = (y_1, \ldots, y_d)^{\mathrm{T}}$ is the given observed image on Ω.
- $\mathbf{n} = (n_1, \ldots, n_d)^{\mathrm{T}}$ is i.i.d. Gaussian noise with variance σ^2.
- The inpainting mask Ω^c is known in advance.
- Goal: recover the unknown true image $\mathbf{x}$ by restoring missing pixels of $\mathbf{x}$ outside Ω and suppress its noise on Ω.
- Solve the inpainting problem $\mathbf{y} = \chi_\Omega \mathbf{x} + \mathbf{n}$ through the minimization scheme:

$$\min_{\mathbf{c} \in \mathbb{R}^n} \frac{1}{2} \|\chi_\Omega \mathcal{D} \mathbf{c} - \mathbf{y}\|_2^2 + \lambda \|\mathbf{c}\|_1 + \kappa \|(I - \mathcal{D}^T \mathcal{D})\mathbf{c}\|_2^2,$$

  where $\mathcal{D} \in \mathbb{R}^{n \times d}$ is a tight frame satisfying $\mathcal{D}\mathcal{D}^T = I_d$ and a reconstructed image is given by $\mathbf{x} = \mathcal{D}\mathbf{c}$.

Random Missing Pixels or Corrupted by Texts

Figure: 80% missing pixels. Recovered by our algorithm: PSNR=31.67.

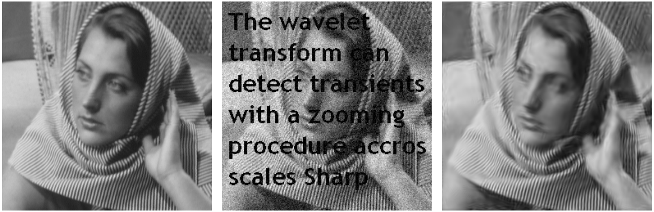


Figure: Corrupted by text with $\sigma = 20$. Recovered with PSNR= 28.93.

Image Inpainting Algorithm

Initialization: $\boldsymbol{x}_0 = 0$, $\lambda = \lambda_0$, $\ell = 0$.
while not convergent **do**
  $\boldsymbol{c}_{\ell+1} = \text{Thresholding}_\lambda(\mathcal{V}^\mathsf{T}(\chi_\Omega \boldsymbol{y} + (I - \chi_\Omega)\boldsymbol{x}_\ell))$.
  $\boldsymbol{x}_{\ell+1} = \mathcal{V}\boldsymbol{c}_{\ell+1}$.
  error $= \|(I - \chi_\Omega) - -(\boldsymbol{x}_\ell - \boldsymbol{x}_{\ell+1})\|_2 / \|\boldsymbol{y}\|_2$.
  if error $<$ tolerence **then**
    Update the thresholding value λ.
  end if
  $\ell = \ell + 1$.
end while
return $\boldsymbol{x}_{\ell+1}$.

Remove Mixed Gaussian and Impulse Noises

Gaussian and Pepper–and–Salt impulse noise. Cameraman: $\sigma = 0$, $p = 0.3$, PSNR $= 32.50$. Lena: $\sigma = 15$, $p = 0.5$, PSNR $= 30.95$.

Gaussian and Random-valued impulse noises: Barbara: $\sigma = 30$, $p = 0.2$, PSNR $= 25.93$. Peppers: $\sigma = 20$, $p = 0.1$, PSNR $= 27.31$.