IUSEP, Mathematical Finance, Project 2

In this project, we study a way to determine the risk-free interest rate, similarly to the implied volatility. We consider a continuous-time model, where S_t is the stock price at time $t \in [0,T]$ and the bank account has value $\mathrm{e}^{\rho t}$ at time t. A (European) put option with strike price K and maturity T is a financial derivative with payoff $\max\{K - S_T, 0\}$ at time T.

1. Show that the following relation holds:
$$\max\{K - S_T, 0\} = \max\{S_T - K, 0\} - S_T + K. \tag{1}$$

2. Assume that a call option with strike price K and maturity T is replicable by a hedging strategy (Δ_t^c, Ψ_t^c) which consists of Δ_t^c units of the stock and Ψ_t^c units of the bank account at time $t \in [0,T]$. Find a hedging strategy (Δ_t^p, Ψ_t^p) for the put option in terms of Δ_t^c and Ψ_t^c.

3. Derive the relation
$$p = c - S_0 + K\mathrm{e}^{-\rho T}, \tag{2}$$
where p and c are the prices of the put and call options, respectively.

4. If a stock pays dividends, one needs to adjust (2) to
$$p = c - S_0\mathrm{e}^{-dT} + K\mathrm{e}^{-\rho T}, \tag{3}$$
where d is the dividend yield (annualized and continuously compounded) and maturity T is measured in years. Use (3) to write a MATLAB function, which returns the interest rate ρ based on suitable input variables.

5. If one calculated the interest rate by applying your MATLAB function from the previous question to option price data for different strike prices (plotting strike prices on the x-axis and the computed interest rates on the y-axis), should one expect a phenomenon analogous to the volatility smile/skew?

6. Use the following data (in USD): On May 10, 2024, the S&P 500 index closed at 5,222. A (European) call option with strike price 5,200 and expiration date July 15, 2024 was traded at 142, and the corresponding put option at 86. The dividend yield on S&P 500 is 1.54 % (source: Standard & Poor's). First determine ρ and then calculate the implied volatilities for the call and put options.