Dynamical Systems on Networks: Part II

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Outline

- A network as a directed graph, examples
- Dynamical systems on networks, examples
- Global-stability problems for network dynamics
- Kirchhoff Matrix-Tree Theorem
- A general global stability result
- Application I: flight formation control of drones
- Application II: global synchronization of coupled oscillators

A Network as a Directed Graph

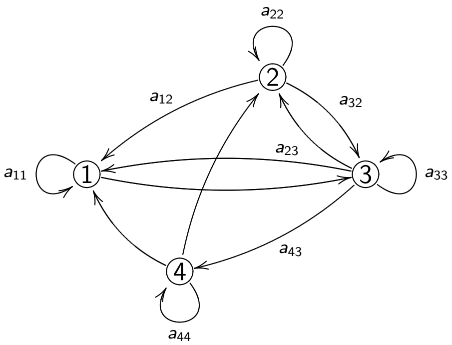
A directed graph $\mathcal{G} = (V, E, A)$

Vertex set: $V = \{1, 2, \cdots, n\}$

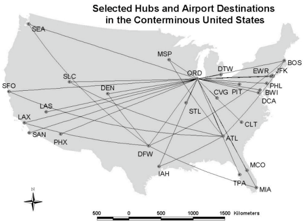
Directed edge: (i, j) from vertex i to j

Weights: $A = (a_{ij}), \ a_{ij} \neq 0 \iff (j, i)$ exists.

Given a nonnegative matrix, there corresponds a digraph $\mathcal{G}_A$, for which A is the weight matrix.



Examples of Networks



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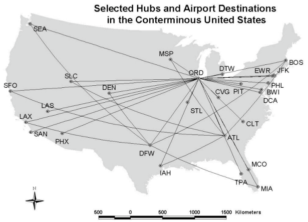

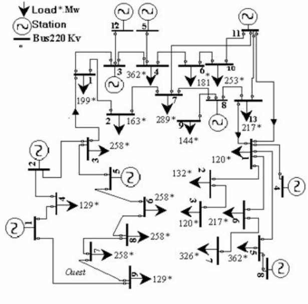

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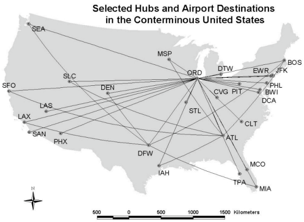

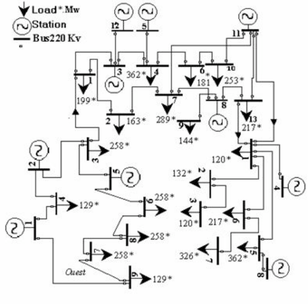

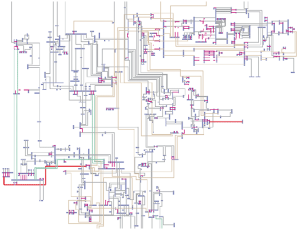

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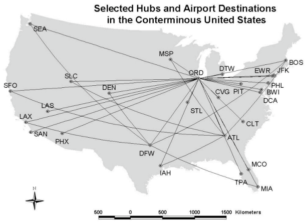

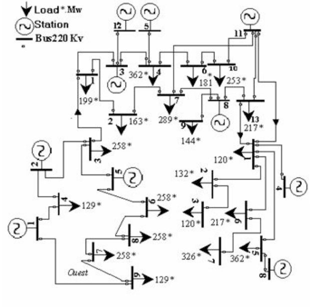

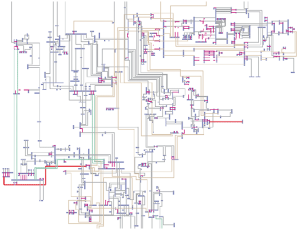

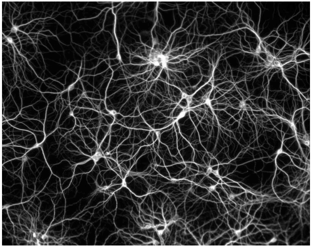

Dynamical Systems on Networks

Given a digraph $G = (V, E)$, a dynamical system can be defined over G.

Vertex dynamics: $u_i' = f_i(t, u_i), \; i = 1, \cdots, n.$
$u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R} \times \mathbb{R}^{m_i} \to \mathbb{R}^{m_i}$.

Connections: $g_{ij} : \mathbb{R} \times \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \to \mathbb{R}^{m_i}$ influence of j on i
$g_{ij} \equiv 0 \iff (j, i)$ does not exist.

Coupled system over G:

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \quad i = 1, 2, \ldots, n.$$

Examples of Dynamical Systems on Networks

- **Coupled Oscillators:**

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0,$$

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- **Dispersal of a single species among n patches**

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \qquad i = 1, 2, \ldots, n.$$

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- **An n-patch predator-prey model**

$$\begin{aligned} x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij}x_i), \\ y_i' &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \end{aligned} \qquad i = 1, 2, \ldots, n.$$

Examples of Dynamical Systems on Networks cont'ed

- **Cellular Neural Network and Lattice Dynamical Systems**

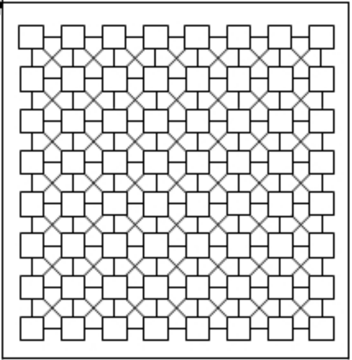

Figure 1: Cellular Neural Network 2D example

Examples of Dynamical Systems on Networks cont'ed

- **A Delayed Hopfield-Cohen-Grossberg Model of Neural Networks**

$$\frac{du_i(t)}{dt} = -u_i(t) + \sum_{i=1}^{n} J_{ij} f\left(u_j(t-\tau)\right), \quad 1 \leq i \leq n$$

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- **An Epidemic Model in Heterogeneous Populations**

$$\begin{aligned} S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j), \\ E_i' &= \sum_{j=1}^{n} \beta_{ij} f_{ij}(S_i, I_j) - (d_i^E + \epsilon_i) E_i, \quad i = 1, 2, \cdots, n. \\ I_i' &= \epsilon_i E_i - (d_i^I + \gamma_i) I_i. \end{aligned}$$

Research Questions

Assume: Independent vertex dynamics are simple or identical

Investigate: If, what, how complex dynamic behaviours emerge through network interactions.

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- Stability and control

Global Stability in Network Dynamics

Given a coupled system over a digraph G:

$$u_i' = f_i(t, u_i) + \sum_{j=1}^{n} g_{ij}(t, u_i, u_j), \quad i = 1, 2, \ldots, n. \tag{1}$$

Assume: Each vertex $u_i' = f_i(t, u_i)$ is globally stable, as insured by a global Lyapunov function V_i.

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Assume: Each vertex $u_i' = f_i(t, u_i)$ is globally stable, as insured by a global Lyapunov function V_i.

Question: Under what conditions on the underlying network and coupling is the coupled system globally stable?

Of significance in disease control, stability of eco-systems, power distribution grids etc.

Main Result

Theorem [Z. Shuai and ML, 2009] Assume

(1) There exist $F_{ij}(t, u_i, u_j)$ such that

$$\dot{V}_i(u) \leq \sum_{j=1}^{n} a_{ij} F_{ij}(t, u_i, u_j), \quad t > 0,\ u_i \in D_i,\ u_j \in D_j, j = 1, \cdots, n. \tag{2}$$

(2) Along each directed cycle $\mathcal{C}$ of G,

$$\sum_{(r,s)\in E(\mathcal{C})} F_{rs}(t, u_r, u_s) \leq 0, \quad t > 0,\ u_r \in D_r,\ u_s \in D_s. \tag{3}$$

Then there exist constants $c_i \geq 0$ such that $V(u) = \sum_{i=1}^{n} c_i V_i(u)$ satisfies

$$\dot{V}(u) \leq 0, \quad u \in D_1 \times \cdots \times D_n.$$

Kirchhoff Matrix-Tree Theorem

Let (G, A) be a weighted digraph with weight matrix $A = (a_{ij})$. The Laplacian matrix of graph G is

$$L = \begin{bmatrix} \sum_{k \neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum_{k \neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum_{k \neq n} a_{nk} \end{bmatrix}.$$

Let c_i be the cofactor of the i-th diagonal element of L.

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Theorem [Kirchhoff (1847)] Assume $n \geq 2$. Then

$$c_i = \sum_{\mathcal{T}\in\mathbb{T}_i} w(\mathcal{T}), \qquad i = 1, 2, \ldots, n, \tag{4}$$

where $\mathbb{T}_i$ is the set of all spanning trees $\mathcal{T}$ of $(\mathcal{G}, A)$ rooted at vertex i, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$.

Reordering of a Double Sum

Proposition [Tree-Cycle-Identity, Z. Shuai and ML 2009] Let c_i be given by the Matrix-Tree Theorem. Then the following identity holds.

$$\sum_{i,j=1}^{n} c_i\, a_{ij}\, F_{ij}(x_i, x_j) = \sum_{\mathcal{Q}\in\mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s)\in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(x_r, x_s), \qquad (5)$$

where $F_{ij}(x_i, x_j), 1 \le i, j \le n$, are arbitrary functions, $\mathbb{Q}$ is the set of all spanning unicyclic graphs $\mathcal{Q}$ of $(\mathcal{G}, A)$, $w(\mathcal{Q})$ is the weight of $\mathcal{Q}$, and $\mathcal{C}_{\mathcal{Q}}$ denotes the oriented cycle of $\mathcal{Q}$.

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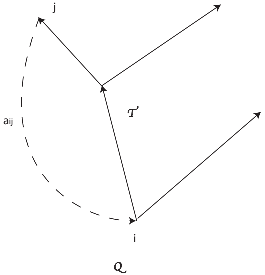
Proof: Note $w(\mathcal{T})\, a_{ij} = w(\mathcal{Q})$,

where $\mathcal{Q}$ is the unicyclic graph obtained by adding an arc (j, i) to $\mathcal{T}$.



Proof of Main Theorem

$$\dot{V} = \sum_{i=1}^{n} c_i \dot{V}_i \leq \sum_{i,j=1}^{n} c_i a_{ij} F_{ij}(t, u_i, u_j) \quad \text{(assumption (1))}$$

$$= \sum_{\mathcal{Q} \in \mathbb{Q}} w(\mathcal{Q}) \sum_{(r,s) \in E(\mathcal{C}_{\mathcal{Q}})} F_{rs}(t, u_r, u_s) \quad \text{(Proposition)}$$

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Our Theorem offers a systematic way to construct global Lyapunov functions for the coupled system, using individual Lyapunov functions for the vertex dynamics.

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Our Theorem offers a systematic way to construct global Lyapunov functions for the coupled system, using individual Lyapunov functions for the vertex dynamics.

Is the theorem any good?

Application I: A Network of Coupled Oscillators

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0, \tag{6}$$

or in systems

$$\begin{aligned} \dot{x}_i &= y_i, \\ \dot{y}_i &= -\alpha_i y_i - f_i(x_i) - \sum_{j=1}^{n} \epsilon_{ij}(y_i - y_j). \end{aligned} \tag{7}$$

Each vertex dynamics is given by a damped nonlinear oscillator

$$\ddot{x}_i + \alpha \dot{x}_i + f_i(x_i) = 0.$$

Assume that the damping $\alpha_i \geq 0$ and the potential energy $F_i(x_i) = \int^{x_i} f_i(s) ds$ has a strictly global minimum at $x_i = x_i^*$. Then $x = x_i^*$ is globally stable (using the Lyapunov function)

$$V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}.$$

Application I: A Network of Coupled Oscillators

Theorem Assume $\alpha_k > 0$ for some k and digraph $\mathcal{G}$ is strongly connected. Then $E^*(x_1^*, 0, x_2^*, 0, \cdots, x_n^*, 0)$ is globally asymptotically stable in $\mathbb{R}^{2n}$.

Proof. $V_i(x_i, y_i) = F_i(x_i) + \frac{y_i^2}{2}$

$$\begin{aligned}\dot{V}_i &= -\alpha_i y_i^2 - \sum_{j=1}^{n} \epsilon_{ij}(y_i - y_j)y_i \\ &\leq \sum_{j=1}^{n} \epsilon_{ij}[-\frac{1}{2}(y_i - y_j)^2 - \frac{1}{2}y_i^2 + \frac{1}{2}y_j^2] \\ &\leq \sum_{j=1}^{n} \epsilon_{ij} F_{ij}(y_i, y_j)\end{aligned}$$

where

$$F_{ij}(y_i, y_j) = -\frac{1}{2}y_i^2 + \frac{1}{2}y_j^2.$$

Application II: A Single Species Model with Dispersal

$$x_i' = x_i f_i(x_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \quad i = 1, 2, \ldots, n. \tag{8}$$

Application II: A Single Species Model with Dispersal

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Theorem [Z. Shuai and ML (2009)] Assume

(1) matrix (d_{ij}) is irreducible;

(2) $f_i'(x_i) \leq 0, x_i > 0, i = 1, 2, \ldots, n;\ \exists\ k,\ f_k'(x_k) \not\equiv 0$ in any open interval of $\mathbb{R}^+$;

(3) system (8) is uniformly persistent;

(4) solutions of (8) are uniformly bounded.

Then system (8) has a globally asymptotically stable positive equilibrium E^*.

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Note: Lu and Tacheuchi (1993) proved the result under the assumption $f_i'(x_i) < 0,\ x_i > 0$ for all i, using the theory of monotone dynamical systems.

Application III: An n-Patch Predator-Prey Model

$$\begin{aligned} x_i' &= x_i(r_i - b_i x_i - e_i y_i) + \sum_{j=1}^{n} d_{ij}(x_j - \alpha_{ij} x_i), \\ y_i' &= y_i(-\gamma_i - \delta_i y_i + \epsilon_i x_i), \end{aligned} \qquad i = 1, 2, \ldots, n. \tag{9}$$

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Theorem [Z. Shuai and ML (2009)] Assume that (d_{ij}) is irreducible, and that $\exists\ k$ such that $b_k \delta_k > 0$. Then the positive equilibrium E^*, whenever it exists, is unique and globally asymptotically stable in the positive cone $\mathbb{R}^{2n}_+$.

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Kuang and Tacheuchi (1994) proved the two-patch case.

$$V_i(x_i, y_i) = \epsilon_i(x_i - x_i^* \ln x_i) + e_i(y_i - y_i^* \ln y_i)$$

Application IV: A Multi-group Delayed Epidemic Model

$$
\begin{aligned}
S_i' &= \Lambda_i - d_i^S S_i - \sum_{j=1}^{n} \beta_{ij} S_i I_j(t-\tau_j), \\
I_i' &= \sum_{j=1}^{n} \beta_{ij} S_i I_j(t-\tau_j) - (d_i^I + \gamma_i) I_i,
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When $n = 1$, C. McCluskey proved the global stability with Lyapunov function

$$V_i = (S_i - S_i^* + S_i^* \ln \frac{S_i}{S_i^*}) + (I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}) +$$

$$\sum_{j=1}^{n} \beta_{ji} S_i^* \int_0^{\tau_j} \left(I_j(t-r) - I_j^* - I_j^* \ln \frac{I_j(t-r)}{I_j^*} \right) dr.$$

Synchronization

Synchronization of metronomes: a video

`https://www.youtube.com/watch?v=Aaxw4zbULMs`

Coupled Oscillators Revisited

Consider a system of coupled oscillators:

$$\ddot{x}_i + f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(\dot{x}_i - \dot{x}_j) = 0,$$

Assume that $f_i(x_i)$ and $F_i(x_i) = \int_i^x f_i(t)dt$ satisfy

(C_1) $f_i(x_i)x_i > 0,\ x_i \neq 0, i = 1, 2, \cdots, n,$

(C_2) $F_i(x_i) \to \infty$ as $|x_i| \to \infty,\ i = 1, 2, \cdots, n.$

Both (C_1) and (C_2) are satisfied for $f_i(x_i) = x_i^3$.

Global Synchronization

Definition: System (21) is said to achieve global synchronization if, for every solution $x(t)$ of system (21) and all $1 \leq i, j \leq n$,

$$\dot{x}_i(t) - \dot{x}_j(t) = 0.$$

Question: Under what conditions of matrix $A = (a_{ij})$ does the system (21) achieves global synchronization?

A Theorem

Theorem (P. Du and ML 2015)

In system (21), suppose that the direct graph $\mathcal{G}_A$ is strongly connected, and assumptions (C_1) and (C_2) are satisfied. Then system (21) achieves global synchronization.

For the proof, considering the equivalent system

$$\dot{x}_i = y_i$$

$$\dot{y}_i = -f_i(x_i) + \sum_{j=1}^{n} \epsilon_{ij}(y_j - y_i)$$

Using Lyapunov functions:

$$V_i = \frac{1}{2}y_i^2 + F_i(x_i),$$

and

$$V = \sum_{i=1}^{n} c_i V_i.$$

Proof

The vertex Lyapunov function is the total energy function

$$V_i(x_i, y_i) = \frac{1}{2}y_i^2 + F_i(x_i), \quad F_i(x_i) = \int_0^{x_i} x_i dx_i$$

Consider a Lyapunov function for the coupled system

$$V(x, y) = \sum_i c_i V_i(x_i, y_i).$$

We verify the V_i satisfies the assumptions of the Li-Shuai Theorem:

$$\dot{V}_i = -f_i(x_i)y_i + \sum_j k_{ij}(y_j - y_i)y_i + f_i(x_i) = \sum_j k_{ij}(y_j - y_i)y_i$$
$$\leq \frac{1}{2}\sum_j k_{ij}[-(y_j - y_i)^2 + y_j^2 - y_i^2] \leq \frac{1}{2}\sum_j k_{ij}(y_j^2 - y_i^2)$$

Li-Shuai Theorem implies that $\dot{V}(x, y) \leq 0$ and $\dot{V} = 0$ if and only if $y_1 = y_2 = \cdots = y_n$. The conclusion then follows from the LaSalle Invariance Principle.

A video on Youtube

https://www.youtube.com/watch?v=QmWD76jwjbQ

GRASP Lab, University of Pennsylvania

Each robotic agent has position vector $r_i = (x_i, y_i) \in \mathbb{R}^2$ and velocity vector $v_i = \dot{r}_i = (\dot{x}_i, \dot{y}_i)$.

The system's evolution is governed by Newton's equation

$$\begin{aligned} \dot{r}_i &= v_i, \\ \dot{v}_i &= u_i, \end{aligned} \qquad i = 1, \cdots, n. \tag{11}$$

Here

- $u_i, \; i = 1, \cdots, n$, define the control protocol
- Formation control is achieved through communications among agents
- Network represents the communication graph (topology)
- A complete communication graph is too costly.

Formation Stabilization Problem

Definition A control protocol is said to solve the formation stabilization problem if solutions of (11) converge asymptotically to a state such that

(a) the relative positions of each agent (i, j) within a cluster are such that a local minimum of the total vertex potential P_{ij} is achieved,

(b) the headings of any two agents (i, j) and (h, k) satisfy $\theta_{ij} = \theta_{hk}$.

Gradient Systems and Potential Functions

Let $x \in \mathbb{R}^n$ and $x \mapsto V(x) \in \mathbb{R}$ be a scalar-valued function that is Lipchitz continuous. A gradient system has the form

$$x'(t) = -\nabla V(x(t)), \quad x \in D \subset \mathbb{R}^n,$$

with the potential function $V(x)$.

Unique characteristics of a gradient system: $V(x(t)$ decreases at the rate of steepest decent.

$$V(x(t))' = \nabla V(x(t)) \cdot x'(t) = \nabla V(x(t)) \cdot [-\nabla V(x(t))] = -||\nabla V(x(t))\|^2,$$

and the gradient $-\nabla V(x)$ points to the direction of the steepest decent and the maximum speed of decent is $||\nabla V(x(t))\|^2$.

Gradient Systems and Potential Functions

Let $\bar{x} \in D \subset \mathbb{R}^n$ and $V(x) = ||x - \bar{x}||^2$. Consider the gradient system

$$x' = -\nabla V(x), \quad x \in D.$$

Then from the preceding discussion we know that:

(1) $\bar{x}$ is an equilibrium

(2) All other solutions $x(t)$ satisfy $||x(t) - \bar{x}|| \to 0$ as $t \to \infty$.

This illustrates how to use the gradient of appropriate potential functions as control functions.

Hierarchical Potential Clustering (HPC) Protocol

Proposed by J. Maidens and ML:

1) Divide the agents into clusters
2) Assign a leader to each cluster
3) Implement an artificial potential scheme (with a complete graph) within each cluster
4) Implement a velocity consensus scheme among the cluster leaders.

HPC Protocol: control within a cluster i

For $j \neq 1$, (i.e. r_{ij} is not a leader in cluster i)

$$u_{ij} = -\nabla_{r_{ij}} P_{ij} - \sum_{k} \frac{(\theta_{ij} - \theta_{ik})||v_{ij}||}{||r_{ij} - r_{ik}||} \hat{n}(ij),$$

where

$$P_{ij} = \sum_{k=1}^{n_i} P_{ij}^{ik}$$

controls distance of agents in the cluster and

$$\theta_{ij} = \tan^{-1}\left(\frac{\dot{y}_{ij}}{\dot{x}_{ij}}\right)$$

is the heading of agent (i, j), and $\hat{n}(ij) \cdot \theta_{ij} = 0$.

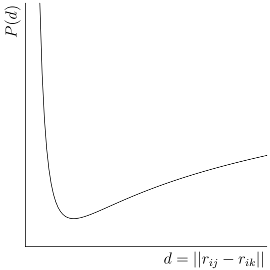

Figure: Potential function P_{ij}^{ik}.

HPC Protocol: control among leaders

For $j = 1$, (i.e., r_{i1} is the leader in cluster i), we add additional force to control there heading

$$u_{i1} = -\nabla_{r_{i1}} P_{i1} - \sum_k \frac{(\theta_{i1} - \theta_{ik})||v_{i1}||}{||r_{i1} - r_{ik}||} \hat{n}(i1) + \sum_{h \in N_i} b_{ih}(v_{h1} - v_{i1})$$

where matrix $B = (b_{ij})$ is any nonnegative irreducible matrix. The correspond communication graph $\mathcal{G}_B$ among leaders is strongly connected.

Main Result

Theorem (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph $\mathcal{G}_B$ is strongly connected.

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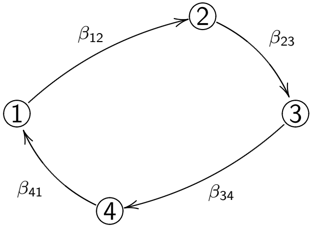
Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph $\mathcal{G}_B$ is strongly connected.

An example graph that is strongly connected:

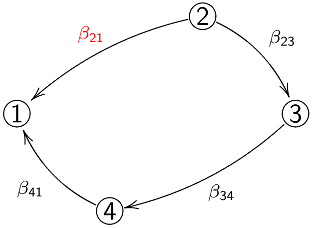


Main Result

Theorem (J. Maidens and ML, 2013)

Given any clustering scheme, the HPC protocol solves the formation stabilization problem provided that the leader communication graph $\mathcal{G}_B$ is strongly connected.

An example graph that is not strongly connected:



Simulations

- Clustering without control protocol
  - Video 1
- Clustering without leader control
  - Video 2
  - Video 3
- Clustering with leader control
  - Video 4
  - Video 5