Online lectures for Math 348: Differential geometry of curves and surfaces

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Lecture 1: What is a curve?

Examples of curves

1. Line in $\mathbb{R}^2$; e.g., $y = mx + b$.
2. Graph in $\mathbb{R}^2$; e.g., $y = x^2$.
3. Level curve of a function in $\mathbb{R}^3$:
   $\{(x, y, z) \big| x^2 + y^2 + z^2 = 1\} \cap \{(x, y, z) \big| z = \frac{1}{2}\}$
4. Curves of intersection;
   e.g.; intersect paraboloid $z = x^2 + y^2$
   with plane $z = \frac{1}{2}y + 1$.
5. Level sets are curves of intersection
   of graphical surfaces $z = f(x, y)$ with planes $z = k$.
   The constant k is the *level*.

Graphs aren't everything

- Circle $x^2 + y^2 = a^2$ is not the graph of a function:
  It can be "double-valued".
- Inelegant solution:
  It's the union of graphs of *two* functions:
  $y = \sqrt{a^2 - x^2}$ and $y = -\sqrt{a^2 - x^2}$, $-a \leq x \leq a$.
- Better solution: Parametrize the circle:
  $x(\theta) = a\cos\theta$
  $y(\theta) = a\sin\theta$
  $\theta \in [0, 2\pi)$

Parametrized curves

Definition

A *parametrized curve* is a map $\gamma : I \to \mathbb{R}^n$, where I is a connected interval of $\mathbb{R}$.

The textbook takes I to be open, because we need to define differentiation. But sometimes we will need endpoints, and then I should be closed or half-closed (see the last slide, where $\theta \in [0, 2\pi)$). We won't impose that I is always open, but will instead assume that any differentiation applies only in the interior of I, or applies in a one-sided sense at endpoints.

- It's very easy to parametrize a graph $y = f(x)$.
- Just choose x to *be* the parameter; i.e., write $x(t) = t$.
- Then $y(t) = f(t)$.
- Don't forget to choose domain (e.g., perhaps $t \in (-\infty, \infty)$, perhaps not).

Examples

- The parametrized curve $\begin{cases} x(t) = & t\ , \\ y(t) = & \sqrt{a^2 - t^2}\ , \\ t \in & [-a, a]\ , \end{cases}$ is a semi-circle.
- The parametrized curve $\begin{cases} x(t) = & \cos t\ , \\ y(t) = & \sin t\ , \\ t \in & [0, 2\pi)\ , \end{cases}$ is a circle, traversed once counter-clockwise.
- The parametrized curve $\begin{cases} x(t) = & \cos t\ , \\ y(t) = & \sin t\ , \\ t \in & [0, 4\pi)\ , \end{cases}$ is a circle, traversed twice counter-clockwise.

Notice the parametrization carries extra information not available from the graphical description of a curve.

Example: The astroid

- The parametrized curve $\gamma(t) = \left(\cos^3 t, \sin^3 t\right)$, $t \in [0, 2\pi)$, is called an *astroid*.
- Can write it as $\begin{cases} x(t) = \cos^3 t \\ y(t) = sin^3 t \\ t \in [0, 2\pi) \end{cases}$
- Then $x^{2/3} = \cos^2 t$ and $y^{2/3} = \sin^2 t$, so $x^{2/3} + y^{2/3} = 1$.
- Graphical form: $y = \pm\left(1 - x^{2/3}\right)^{3/2}$.
- Level set form:
  - Let $z = f(x, y) = x^{2/3} + y^{2/3}$.
  - Then the astroid is the level set $z = f(x, y) = 1$.
- Graphical and level set forms have less information than parametrized form, but produce the same image. The image of a curve is called the *trace* of the curve (not related to the trace of a matrix).

Tangent vectors

- Recall tangent line to graph $y = f(x)$ at (x_0, y_0) is $y - y_0 = f'(x_0)\,(x - x_0)$.
- Tangent vector: Any (non-zero) vector parallel to tangent line.
- Parametrized form of line: Take $s \in \mathbb{R}$ and
  $x(s) = x_0 + s$
  $y(s) = y_0 + f'(x_0)s$
- Differentiate wrt s: $x'(s) = 1$, $y'(s) = f'(x_0)$.
- Tangent vectors to line are the vectors parallel to $(1, f'(x_0))$.

Tangent vectors to parametrized curves

- Parametrized curve $\gamma : I \to \mathbb{R}^n$ is a *vector-valued* function.
- $\gamma(t) = (\gamma_1(t), \gamma_x(t), \dots, \gamma_n(t)) = (x_1(t), x_2(t), \dots, x_n(t))$.

Definition

$$\gamma'(t) = \dot{\gamma}(t) = \frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}, \dots, \frac{d\gamma_n}{dt} \right)$$
$$= \lim_{\Delta t \to 0} \frac{\gamma(t + \Delta t) - \gamma(t)}{\Delta t}$$

Then $\gamma'(t)$ is a tangent vector to curve γ at t provided $\gamma'(t) \neq (0, \dots, 0)$,

(Generally, we will just write 0 even if we mean the 0-vector $(0, 0, \dots, 0)$.)

Example

- $\gamma(t) = t^3\mathbf{e}_1 + t^2\mathbf{e}_2,\ t \in \mathbb{R}$,
  $\{\mathbf{e}_1, \mathbf{e}_2\}$ = orthonormal basis (ONB).
- $\left.\begin{array}{l}\gamma_1(t) = t^3 \\ \gamma_2(t) = t^2\end{array}\right\} \implies y = x^{2/3}$.
- Chain rule:
  - $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

$\implies \quad 2t = \frac{dy}{dx} \cdot 3t^2$

$\implies \quad \frac{dy}{dx} = \frac{2t}{3t^2}$ undefined at $t = 0$.

Definition

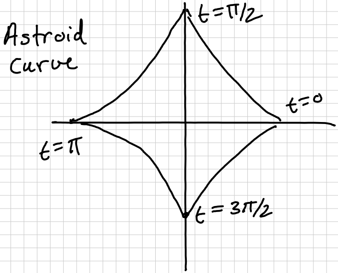
A parametrized curve $\gamma : I \to \mathbb{R}^n$ is

- *smooth* at $t_0 \in I$ if all derivatives of all components $\gamma_i(t)$ exist at $t = t_0$, and
- *regular* at $t_0 \in I$ if it is smooth at t_0 and $\frac{d\gamma}{dt}(t_0) \neq (0, \ldots, 0)$; otherwise t_0 is a *singular point*.

The above example is smooth but not regular at $t = 0$

The astroid again

- $\gamma(t) = \left(\cos^3 t, \sin^3 t\right)$, and say $t \in [0, 2\pi)$.
- Differentiate: $\gamma'(t) = \left(-3\sin t \cos^2 t, 3\sin^2 t \cos t\right)$, $t \in [0, 2\pi)$.
- Simplify: $\gamma'(t) = 3\cos t \sin t\,(-\cos t, \sin t)$, $t \in [0, 2\pi)$.
- Then $\gamma'(t) = 0 \Leftrightarrow \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
- Therefore γ is smooth everywhere, but it is not regular at four points.



Lecture 2: Arclength and tangent vectors

Arclength

- Recall arclength in $\mathbb{R}^2$:
$$s = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{d\gamma_1^2 + d\gamma_2^2} = \int\limits_{t_0}^{t_1} \sqrt{\left(\frac{d\gamma_1}{dt}\right)^2 + \left(\frac{d\gamma_2}{dt}\right)^2} dt$$
- In $\mathbb{R}^n$: $s = \int\limits_{t_0}^{t_1} \sqrt{\left(\frac{d\gamma_1}{dt}\right)^2 + \cdots + \left(\frac{d\gamma_n}{dt}\right)^2} dt = \int\limits_{t_0}^{t_1} \sqrt{\frac{d\gamma}{dt} \cdot \frac{d\gamma}{dt}} dt = \int\limits_{t_0}^{t_1} \left\| \frac{d\gamma}{dt} \right\| dt$

Definition

The arclength function of a curve $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is
$s := \int\limits_{t_0}^{t} \left\| \frac{d\gamma(t')}{dt'} \right\| dt'$
for $t \in [t_0, t_1]$.

Fundamental Theorem of Calculus $\implies \frac{ds}{dt} = \left\| \frac{d\gamma(t)}{dt} \right\|$.
This is called the *speed* of the curve.

Example: Log spiral

- The logarithmic spiral is the curve $\gamma(t) = e^t\,(\cos t, \sin t)$.
- $\gamma'(t) = e^t\,(\cos t - \sin t, \sin t + \cos t)$
- $\|\gamma'\| = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2} = \sqrt{2}e^t$.
- $s(t) = \int\limits_{t_0}^{t} \sqrt{2}e^\tau\,d\tau = \sqrt{2}\,(e^t - e^{t_0})$.
- $t_0 \to -\infty \implies \gamma(t_0) \to (0,0),\ s(t) \to \sqrt{2}e^t$.
- $\gamma : (-\infty, t] \to \mathbb{R}^2$ has finite arclength, but no initial endpoint.

Unit speed curves

- If $\|\dot{\gamma}(t)\| = 1$, γ is *unit speed* and t is an *arclength parameter* or *unit speed parameter*.
- If $\|\dot{\gamma}(t)\| = k = const > 0$, γ is *constant speed* and t is an *affine parameter*.
- Fact:
  - Let $\mathbf{v}$ be any unit vector field $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1$.
  - Let $\gamma(t)$ be a unit speed curve.
  - $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \frac{d}{dt}(1) = 0$.
  - But then $\frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = 0$.
  - Chain rule: $\dot{\gamma} \cdot \ddot{\gamma} = 0$.
  - Conclude that $\dot{\gamma} \perp \ddot{\gamma}$ along any unit speed curve whenever acceleration $\ddot{\gamma} \neq 0$.
  - For unit speed curves, write $\mathbf{t} := \dot{\gamma} =$ unit tangent vector. Note that $\|\mathbf{t}\| = \sqrt{\mathbf{t} \cdot \mathbf{t}} = 1$.

Reparametrization

- Say $\gamma : (a, b) \to \mathbb{R}^n$ is a curve, and
- Say $\tilde{\gamma} : (\tilde{a}, \tilde{b}) \to \mathbb{R}^n$ is a curve.

Definition

If

- there is a smooth map $\phi : (\tilde{a}, \tilde{b}) \to (a, b)$
- with smooth inverse $\phi^{-1} : (a, b) \to (\tilde{a}, \tilde{b})$, such that
- $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t})) = (\gamma \circ \phi)(\tilde{t}) = \gamma(t)$ for all $\tilde{t} \in (a, b)$,

then $\tilde{\gamma} = \gamma \circ \phi$ is a *reparametrization* of γ.

Theorem

Theorem

Any reparametrization of a regular curve is also a regular curve.

Proof.

- Let $t = \phi(\tilde{t})$ and $\tilde{\gamma}(\tilde{t}) = \gamma(t)$.
- Then $\tilde{t} = \phi^{-1}(t)$ so $t = \phi(\tilde{t}) = \phi(\phi^{-1}(t))$.
- Chain rule: $\frac{d\phi}{d\tilde{t}} \frac{d(\phi^{-1})}{dt} = 1$, so $\frac{d\phi}{d\tilde{t}} \neq 0$.
- $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d}{d\tilde{t}}(\gamma(t)) = \frac{d\gamma}{dt} \frac{d\phi}{d\tilde{t}}$.
- Now γ is regular so $\frac{d\gamma}{dt} \neq 0$, and $\frac{d\phi}{d\tilde{t}} \neq 0$.
- Thus $\frac{d\tilde{\gamma}}{d\tilde{t}} \neq 0$.

□

Works iff reparametrization ϕ is smooth with smooth inverse.

The arclength function of a regular curve is smooth

- Say $\gamma : I \to \mathbb{R}^2 : t \to (x(t), y(t))$ is a regular curve.
- Then $x(t)$ and $y(t)$ are smooth functions.
- The square root function $f(w) = \sqrt{w}$ is smooth if $w \neq 0$.
- Since γ is regular, $\dot{x}^2 + \dot{y}^2 \neq 0$.
- Thus $\frac{ds}{dt}(t) = \sqrt{\dot{x}^2 + \dot{y}^2}$ is smooth.
- Therefore $s(t) = \int\limits_{t_0}^{t} \frac{ds}{dt'}(t')dt'$ is smooth.

Regular curve have unit speed parametrizations

Theorem

A parametrized curve has an arclength parametrization iff it is regular.

Proof.

- Curve $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^2$ and reparametrization $t = \phi(\tilde{t})$, such that $\gamma(t) = \tilde{\gamma}(\tilde{t})$.
- Chain rule: $\frac{d\tilde{\gamma}}{d\tilde{t}} = \frac{d\gamma}{dt}\frac{dt}{d\tilde{t}} \implies \left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = \left\| \frac{d\gamma}{dt} \right\| \left| \frac{dt}{d\tilde{t}} \right|$.

$\Rightarrow$ If $\tilde{t}$ is arclength, then $\left\| \frac{d\tilde{\gamma}}{d\tilde{t}} \right\| = 1$, so $\frac{d\gamma}{dt}$ is never zero. Then $\gamma(t)$ is regular.

$\Leftarrow$
- If $\frac{d\gamma}{dt} \neq 0$, then $\frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\| \neq 0$, so s is smooth and strictly increasing.
- Then $\frac{d\gamma}{dt} = \frac{d\tilde{\gamma}}{ds}\frac{ds}{dt} \implies \left\| \frac{d\gamma}{dt} \right\| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \left| \frac{ds}{dt} \right| = \left\| \frac{d\tilde{\gamma}}{ds} \right\| \frac{ds}{dt}$.
- But $s = \int \left\| \frac{d\gamma}{dt} \right\| dt \implies \frac{ds}{dt} = \left\| \frac{d\gamma}{dt} \right\|$.
- Compare last two lines. Then $\left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1$, so $\tilde{\gamma}(s)$ is unit speed.

Example

Parametrize curve $\gamma(t) = \left(\cos^3 t, \sin^3 t, \cos 2t\right) \in \mathbb{R}^3$, $t \in [0, \pi/2]$ by arclength.

Solution:

- $\dot{\gamma}(t) = \left(-3\cos^2 t \sin t, 3\sin^2 t \cos t, -2\sin 2t\right)$.

$$\begin{aligned} \|\dot{\gamma}\|^2 &= 9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 4\sin^2 2t \\ &= 9\cos^2 t \sin^2 t + 16\cos^2 t \sin^2 t \\ &= 25\cos^2 t \sin^2 t \ . \end{aligned}$$

- Then $\|\dot{\gamma}\| = 5\cos t \sin t$ for $t \in [0, \pi/2]$.
- $s = \int\limits_0^t \|\dot{\gamma}(\tau)\|\, d\tau = 5\int\limits_0^t \cos\tau \sin\tau d\tau = \frac{5}{2}\sin^2 t$.
- Then $\frac{2s}{5} = \sin^2 t$, so $1 - \frac{2s}{5} = \cos^2 t$, and then $\cos 2t = \cos^2 t - \sin^2 t = 1 - \frac{4s}{5}$.
- $\tilde{\gamma}(s) = \left(\left(1 - \frac{2s}{5}\right)^{3/2}, \left(\frac{2s}{5}\right)^{3/2}, 1 - \frac{4s}{5}\right)$.

Regular curve, non-regular parametrization

- Parabola $y = x^2$.
- Regular parametrization $x(t) = t$, $y(t) = t^2$, $t \in \mathbb{R}$.
- Then $\dot{x} = 1$, $\dot{y} = 2t$, and $\dot{x}^2 + \dot{y}^2 = 1 + 4t^2 \neq 0$.
- Non-regular parametrization $x(t) = t^3$, $y(t) = t^6$, $t \in \mathbb{R}$.
- Then $\dot{x} = 3t^2$, $\dot{y} = 6t^5$, and $\dot{x}^2 + \dot{y}^2 = 9t^4 + 36t^{10}$, equals 0 when $t = 0$.
- What went wrong: Reparametrization map $\phi(t) = t^3$ has inverse $\phi^{-1}(t) = t^{1/3}$, which is not differentiable at $t = 0$, so theorem on regular reparametrizations fails.

Closed curves

Example:

- Ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$, $p, q > 0$ are constants.
- Parametrize as $\gamma(t) = (p \cos t, q \sin t)$, $t \in \mathbb{R}$.
- Then $\gamma(t + 2\pi) = \gamma(t)$ for all $t \in \mathbb{R}$.
- γ is 2π-periodic.

Definition

- If $\gamma(t + T) = \gamma(t)$ for all t and for some $T > 0$, then γ is *T-periodic*.
- If $\gamma(t) = p$ for all t (where $p \in \mathbb{R}^n$ is a point), then γ is a *constant curve*.
- If γ is T-periodic and not constant, then γ is a *closed curve*.

Examples

- Ellipses (including circles) are closed curves.
- The curve $\gamma(t) = (t^2 - 1, t^3 - t)$, $t \in \mathbb{R}$, is not closed.
  - Curve has $\gamma(-1) = \gamma(1) = (0, 0)$.
  - But $\gamma(t + T) = \gamma(t)$ with $T = 2$
    is only true when $t = -1$,
    not true for all t.
  - This curve is not closed and not periodic
    but it does have a closed loop.

Lecture 3: Curvature of plane curves

Curvature

When is a curve ...*curved*?

Definition

If $\gamma : I \to \mathbb{R}^n$ is a unit speed curve, then its curvature is $\kappa := \|\ddot{\gamma}\|$.

Interpretation: Curvature as quadratic coefficient in Taylor's theorem:

$$\gamma(t_0 + \Delta t) = \gamma(t_0) + \dot{\gamma}(t_0)\Delta t + \frac{1}{2}\ddot{\gamma}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

- Can replace $\dot{\gamma}(t_0)$ by unit tangent $\mathbf{t}(t_0) = \dot{\gamma}(t_0)$.
- $\dot{\gamma}(t_0) \cdot \dot{\gamma}(t_0) = 1 \implies 2\dot{\gamma}(t_0) \cdot \ddot{\gamma}(t_0) = 0$, so $\ddot{\gamma} \perp \dot{\gamma}$ for a unit speed curve (if $\ddot{\gamma} \neq 0$).
- Then $\ddot{\gamma} = \pm\kappa\mathbf{n}$ where $\mathbf{n}$ is unit normal vector (orthogonal to $\mathbf{t}$).
- Get $\gamma(t_0 + \Delta t) = \gamma(t_0) + \mathbf{t}(t_0)\Delta t \pm \frac{1}{2}\kappa(t_0)\mathbf{n}(t_0)(\Delta t)^2 + \mathcal{O}(\Delta t^3)$
- Two choices for $\mathbf{n}$: we choose it so that $\{\mathbf{t}, \mathbf{n}\}$ is *right-handed*.

Curvature formulas: general parametrization

- Say t is a general parameter for γ, and s is an arclength parameter.
- Chain rule $\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} \implies \frac{d\gamma}{ds} = \frac{d\gamma/dt}{ds/dt}$.
- Chain rule again $\frac{d^2\gamma}{ds^2} = \frac{d}{ds}\left(\frac{d\gamma/dt}{ds/dt}\right) = \frac{dt}{ds}\frac{d}{dt}\left(\frac{d\gamma/dt}{ds/dt}\right) = \frac{\ddot{\gamma}(t)\dot{s}(t)-\dot{\gamma}(t)\ddot{s}(t)}{(\dot{s}(t))^3}$.
- Now use $\kappa = \left\|\frac{d^2\gamma}{ds^2}\right\|$.
- Then $\kappa = \frac{\|\ddot{\gamma}\dot{s}-\dot{\gamma}\ddot{s}\|}{|\dot{s}|^3}$.
- Then $\kappa = \frac{\left\|\ddot{\gamma}\dot{s}^2-\dot{\gamma}\dot{s}\ddot{s}\right\|}{|\dot{s}|^4} = \frac{\|\ddot{\gamma}(\dot{\gamma}\cdot\dot{\gamma})-\dot{\gamma}(\dot{\gamma}\cdot\ddot{\gamma})\|}{\left(\|\dot{\gamma}\|^2\right)^2}$, using that $\dot{s}^2 = \left(\frac{ds}{dt}\right)^2 = \|\dot{\gamma}\|^2 = \dot{\gamma}\cdot\dot{\gamma}$ and therefore $\dot{s}\ddot{s} = \dot{\gamma}\cdot\ddot{\gamma}$.
- Finally, the "BAC-CAB rule" $\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ yields $\kappa = \frac{\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\|}{\|\dot{\gamma}\|^4}$.
- Notice that $\dot{\gamma}\perp\ddot{\gamma}\times\dot{\gamma}$. Thus $\|\dot{\gamma}\times(\ddot{\gamma}\times\dot{\gamma})\| = \|\dot{\gamma}\|\,\|\ddot{\gamma}\times\dot{\gamma}\|$, so $\kappa = \frac{\|\ddot{\gamma}\times\dot{\gamma}\|}{\|\dot{\gamma}\|^3}$.

Example: Circle

- Circle in $\mathbb{R}^2$: $\gamma(t) = (x_0 + a\cos t, y_0 + a\sin t),\ t \in [0, 2\pi)$.
- $\dot\gamma = a(-\sin t, \cos t)$, $\ddot\gamma = -a(\cos t, \sin t)$.
- Use $\kappa = \frac{\|\ddot\gamma \times \dot\gamma\|}{\|\dot\gamma\|^3}$. Think of $\mathbb{R}^2$ as $z = 0$ plane in $\mathbb{R}^3$.
- $$\dot\gamma \times \ddot\gamma = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e}_3 \\ -a\sin t & a\cos t & 0 \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$
  $$= \mathbf{e}_1 \begin{vmatrix} a\cos t & 0 \\ -a\sin t & 0 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} -a\sin t & 0 \\ -a\cos t & 0 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} -a\sin t & a\cos t \\ -a\cos t & -a\sin t \end{vmatrix}$$
  $$= \mathbf{e}_3 \left(a^2\sin^2 t + a^2\cos^2 t\right) = a^2\mathbf{e}_3.$$
- Also, $\|\dot\gamma\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$.
- Then $\kappa = \frac{a^2\|\mathbf{e}_3\|}{a^3} = \frac{1}{a}$. Circles have constant curvature $= 1/\text{radius}$.

Osculating circles

Definition

If a curve $\gamma : I \to \mathbb{R}^2$ has curvature $\kappa(t) \neq 0$ at point $p = \gamma(t)$, we define its *radius of curvature* at p to be $\rho(t) = 1/\kappa(t)$.

The *osculating circle* to γ at p
is the circle that

- passes through p,
- has the same tangent line as γ at p,
- has radius $\rho = \frac{1}{\kappa}$, and
- lies on the concave side of γ.

Signed curvature

- Parametrize the curve $\gamma(t)$ in $\mathbb{R}^2$.
- The direction of increasing parameter is the *orientation*.
- Define the unit tangent vector $\mathbf{t} = \dot{\gamma} / \|\dot{\gamma}\|$.
- Define the unit normal $\mathbf{n}$ by rotating $\mathbf{t}$ by $\frac{\pi}{2}$ *counter-clockwise* (also called the *right-handed sense*).
- Then the *signed curvature* κ_S is defined by

  $$\ddot{\gamma}(s) = \kappa_S \mathbf{n}$$

  where s is an arclength parameter with $ds/dt > 0$ (i.e., same orientation as t).
- Relation to (ordinary) curvature is $\kappa := |\kappa_S|$.

Interpretation: turning angle

Theorem (The turning angle)

There is a unique smooth function ϕ, called the turning angle, along the regular curve γ such that $\phi(s_0) = \phi_0$ and $\mathbf{t} = (\cos\phi(s), \sin\phi(s))$.

- Tangent vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:
  $\mathbf{t} = \dot\gamma(s) = (\cos\phi(s), \sin\phi(s))$
- Calculate: $\dot{\mathbf{t}} = \ddot\gamma(s) = \dot\phi(s)\,(-\sin\phi(s), \cos\phi(s))$
- Normal vector in $\{\mathbf{e}_1, \mathbf{e}_2\}$ basis:
  $\mathbf{n} = \left(\cos\left(\phi(s) + \frac{\pi}{2}\right), \sin\left(\phi(s) + \frac{\pi}{2}\right)\right)$
  $= (-\sin\phi(s), \cos\phi(s))$
- Conclude that $\ddot\gamma(s) = \dot\phi(s)\mathbf{n}$.
- Compare to $\ddot\gamma(s) = \kappa_S\mathbf{n}$ to get $\boxed{\kappa_S(s) = \dot\phi(s).}$
- The signed curvature is the rate of change of the turning angle wrt arclength.

Hopf's Umlaufsatz (rotation rate)

- Integrate $\kappa_S(s) = \dot{\phi}(s)$ over curve γ.
- $\int\limits_{s_0}^{s} \kappa_S(u)du = \int\limits_{s_0}^{s} \dot{\phi}(u)du = \phi(s) - \phi(s_0)$.
- Take γ closed, with period T.
- $\int\limits_{s_0}^{s_0+T} \kappa_S(u)du = \phi(s_0 + T) - \phi(s_0)$.
- But $\phi(s_0 + T) - \phi(s_0) = 2\pi k$, $k \in \mathbb{Z}$.
- In fact, can argue that $k = \pm 1$ if curve traversed once; k is the *winding number*.

Theorem (Hopf's Umlaufsatz)

The total curvature of a closed curve of period T is $\int\limits_{s_0}^{s_0+T} \kappa_S(u)du = \pm 2\pi$.

Lecture 4: Isometries of $\mathbb{R}^n$

Isometries of $\mathbb{R}^n$

Definition (Isometry of $\mathbb{R}^n$)

$F : \mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* of $\mathbb{R}^n$ if it preserves the distance between any two points:

$$\|F(\mathbf{v}) - F(\mathbf{w})\| = \|\mathbf{v} - \mathbf{w}\|$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

Definition (Orthogonal matrix)

An $n \times n$ matrix P is *orthogonal* if its columns (rows) form an orthonormal set of column (row) vectors. Equivalently, the transpose of P is the inverse: $P^T = P^{-1}$. We write $P \in \mathrm{O}(n) =$ the *group* of orthogonal $n \times n$ matrices.

Theorem (All isometries of $\mathbb{R}^n$)

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be given by $F(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$. Here $\mathbf{v}$ and $\mathbf{a}$ are column vectors and P is an $n \times n$ orthogonal matrix. Then F is an isometry, and all isometries of $\mathbb{R}^n$ can be written this way.

Proving that $F(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ is an isometry

Calculate:

$$\begin{aligned}
\|F(\mathbf{v}) - F(\mathbf{w})\|^2 &= (F(\mathbf{v}) - F(\mathbf{w})) \cdot (F(\mathbf{v}) - F(\mathbf{w})) \\
&= [P\mathbf{v} - P\mathbf{w}]^T [P\mathbf{v} - P\mathbf{w}] \\
&= [\mathbf{v} - \mathbf{w}]^T P^T P [\mathbf{v} - \mathbf{w}] \\
&= [\mathbf{v} - \mathbf{w}]^T [\mathbf{v} - \mathbf{w}] \\
&= (\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) \\
&= \|\mathbf{v} - \mathbf{w}\|^2
\end{aligned}$$

- This proves that $F(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ preserves the distance, so F is an isometry.
- Fact: $F^{-1}(\mathbf{w}) = P^T\mathbf{w} - P^T\mathbf{a}$ is also an isometry.

Proving all isometries can be written as $F(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$

- Orthonormal basis $\{\mathbf{e}_i\}$ and vectors $\mathbf{w}_i := F(\mathbf{e}_i) - F(\mathbf{0})$, $i = 1, \dots, n$.
- The $\mathbf{w}_i$ are unit vectors
  - $\|\mathbf{w}_i\| = \|F(\mathbf{e}_i) - F(\mathbf{0})\| = \|\mathbf{e}_i - \mathbf{0}\| = \|\mathbf{e}_i\| = 1$ since F is an isometry.
  - Then $\|\mathbf{w}_i - \mathbf{w}_j\|^2 = \mathbf{w}_i \cdot \mathbf{w}_i + \mathbf{w}_j \cdot \mathbf{w}_j - 2\mathbf{w}_i \cdot \mathbf{w}_j = 2 - 2\mathbf{w}_i \cdot \mathbf{w}_j$.
- But $\|\mathbf{w}_i - \mathbf{w}_j\|^2 = \|F(\mathbf{e}_i) - F(\mathbf{e}_j)\|^2 = \|\mathbf{e}_i - \mathbf{e}_j\|^2 = (\mathbf{e}_i - \mathbf{e}_j) \cdot (\mathbf{e}_i - \mathbf{e}_j)$
  $= \mathbf{e}_i \cdot \mathbf{e}_i + \mathbf{e}_j \cdot \mathbf{e}_j - 2\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 2 & i \neq j \\ 0 & i = j \end{cases}.$
- Compare last two lines to conclude that $\mathbf{w}_i \perp \mathbf{w}_j$ if $i \neq j$.
- Thus $\{\mathbf{w}_i\}$ is an orthonormal basis too, and so $\mathbf{w}_i = P\mathbf{e}_i$ for some $P \in \mathrm{O}(n)$.
- Endgame: Using $\mathbf{w}_i := F(\mathbf{e}_i) - F(\mathbf{0})$ then $F(\mathbf{e}_i) = \mathbf{w}_i + F(\mathbf{0}) = P\mathbf{e}_i + \mathbf{a}$, for some $P \in \mathrm{O}(n)$ and for $\mathbf{a} = F(\mathbf{0})$.
- Finally, if $F(\mathbf{e}_i) = P\mathbf{e}_i + \mathbf{a}$ for basis $\{\mathbf{e}_i\}$, then $F(\mathbf{v}) = P\mathbf{v} + \mathbf{a}$ for all v.

Direct isometries

- $P \in \mathrm{O}(n) \implies P^{-1} = P^T \implies P^T P = \mathbb{I}_n$.
- Then $\det(P^T P) = (\det P)^2 = \det \mathbb{I} = 1$, so $\det P = \pm 1$.
- If $\det P = 1$ the corresponding isometry F is a *direct isometry*.
  - Preserves orientations of basis sets.
  - Includes rotations about the origin $F(\mathbf{v}) = P\mathbf{v}$, and say $P \in \mathrm{SO}(n) =$ *special orthogonal group*.
  - Includes translations $F(\mathbf{v}) = \mathbf{v} + \mathbf{a}$.
  - Every direct isometry in $\mathbb{R}^2$ is a composition of a rotation about the origin and a translation.
  - Every direct isometry in $\mathbb{R}^3$ is a composition of a rotation about an axis through the origin and a translation.
- If $\det P = -1$ the corresponding isometry F is an *opposite isometry*.
  - Reverses orientations of bases.
  - Includes reflections in planes in $\mathbb{R}^3$.

Fundamental theorem of plane curves

Theorem

Let $k : (\alpha, \beta) \to \mathbb{R}$ *be any smooth function.*

- *There is a unit speed curve* $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ *whose signed curvature is* $\kappa_S = k$.
- *If* $\tilde{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ *is another unit speed curve with the same domain and if its signed curvature also equals* k*, then there is a direct isometry* $M : \mathbb{R}^2 \to \mathbb{R}^2$ *such that*

$$\tilde{\gamma}(s) = M(\gamma(s)) = (M \circ \gamma)(s) \text{ for all } s \in (\alpha, \beta).$$

Proof of part 1

- Fix $s_0 \in (\alpha, \beta)$. Given function k, define $\varphi(s) = \int\limits_{s_0}^{s} k(u)du$ and notice that $\dot{\varphi} = k(s)$ by Fundamental Theorem of Calculus (FTC).
- Define curve $\gamma(s) = \left(\int\limits_{s_0}^{s} \cos(\varphi(u))du, \int\limits_{s_0}^{s} \sin(\varphi(u))du \right)$.
- Compute: $\dot{\gamma}(s) = (\cos(\varphi(s)), \sin(\varphi(s)))$ by FTC.
- Clearly $\|\dot{\gamma}\| = \sqrt{\cos^2 \varphi + \sin^2 \varphi} = 1$ for γ is unit speed.
- Also, clearly φ is the turning angle for our curve γ, so we know that $\kappa_S = \dot{\varphi}(s)$.
- But $\dot{\varphi}(s) = k(s)$, so $\kappa_S(s) = k(s)$ which proves part 1.

Proof of part 2

- Two unit speed curves $\gamma, \tilde{\gamma}(\alpha, \beta) \to \mathbb{R}^2$:
- $\dot{\gamma} = (\cos\varphi(s), \sin\varphi(s))$, and say $\varphi(s_0) = 0$.
- $\dot{\tilde{\gamma}} = (\cos\tilde{\varphi}(s), \sin\tilde{\varphi}(s))$, and say $\varphi(s_0) = \tilde{\varphi}_0$.
- Then $\tilde{\gamma}(s) = \left(\int\limits_{s_0}^{s} \cos\tilde{\varphi}(u) du, \int\limits_{s_0}^{s} \sin\tilde{\varphi}(u) du \right) + \tilde{\gamma}(0)$.
- And $k(s) = \dot{\tilde{\varphi}}(s) = \dot{\varphi}(s)$, so $\tilde{\varphi}(s) = \int\limits_{s_0}^{s} k(u) du + \tilde{\varphi}(s_0) = \varphi(s) + \tilde{\varphi}_0$.
- Then $\tilde{\gamma}(s) = \left(\int\limits_{s_0}^{s} \cos\left(\varphi(u) + \tilde{\varphi}_0\right) du, \int\limits_{s_0}^{s} \sin\left(\varphi(u) + \tilde{\varphi}_0\right) du \right) + \tilde{\gamma}(0)$.
- Use $\cos(A+B) = \cos A \cos B - \sin A \sin B$,
  $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

Proof of part 2 continued

$\Longrightarrow$

$$\tilde{\gamma}(s) = \left(\cos\tilde{\varphi}_0 \int\limits_{s_0}^{s} \cos\varphi(u)du - \sin\tilde{\varphi}_0 \int\limits_{s_0}^{s} \sin\varphi(u)du, \right.$$

$$\left. \sin\tilde{\varphi}_0 \int\limits_{s_0}^{s} \cos\varphi(u)du + \cos\tilde{\varphi}_0 \int\limits_{s_0}^{s} \sin\varphi(u)du \right) + \tilde{\gamma}(0)$$

$$= (\gamma_1(s)\cos\tilde{\varphi}_0 - \gamma_2(s)\sin\tilde{\varphi}_0, \gamma_1(s)\sin\tilde{\varphi}_0 + \gamma_2(s)\cos\tilde{\varphi}_0)$$

using $\gamma(s) = (\gamma_1(s), \gamma_2(s)) = \left(\int\limits_{s_0}^{s} \cos\varphi(u)du, \int\limits_{s_0}^{s} \sin\varphi(u)du \right)$.

- Matrix form:

$$\begin{bmatrix} \tilde{\gamma}_1(s) \\ \tilde{\gamma}_2(s) \end{bmatrix} = \begin{bmatrix} \cos\tilde{\varphi}_0 & -\sin\tilde{\varphi}_0 \\ \sin\tilde{\varphi}_0 & \cos\tilde{\varphi}_0 \end{bmatrix} \begin{bmatrix} \gamma_1(s) \\ \gamma_2(s) \end{bmatrix} + \begin{bmatrix} \tilde{\gamma}_1(s_0) \\ \tilde{\gamma}_2(s_0) \end{bmatrix}$$

$$\Longrightarrow \left[\tilde{\gamma}_(s)\right] = [P(\tilde{\varphi}_0)]\,[\gamma(s)] + [\tilde{\gamma}_0]\,.$$

Proof of part 2 continued

- Last slide: $\left[\tilde{\gamma}_(s)\right] = [P(\tilde{\varphi}_0)]\,[\gamma(s)] + [\tilde{\gamma}_0]$, and $[P(\tilde{\varphi}_0)] = \begin{bmatrix} \cos\tilde{\varphi}_0 & -\sin\tilde{\varphi}_0 \\ \sin\tilde{\varphi}_0 & \cos\tilde{\varphi}_0 \end{bmatrix}$. This is a rotation matrix.
- Then $\tilde{\gamma}$ is obtained by applying a rotation $P(\tilde{\varphi}_0)$ through angle $\tilde{\varphi}_0$ and a translation $T(|bfa)$, $\mathbf{a} = \tilde{\gamma}_0$, to γ.
- Since the composition of a rotation and a translation is an isometry of $\mathbb{R}^2$, this proves part 2.

Consequence: Every unit speed curve in $\mathbb{R}^2$ is completely determined by

- choosing one point on the curve,
- choosing the direction of $\mathbf{t}$ at that point, and
- specifying the curvature function $k(s)$.

and any smooth function is the curvature function of some curve.

Example

Theorem

Any regular curve $\gamma : (a, b) \to \mathbb{R}^2$ with constant curvature $\kappa = c > 0$ is (isometric to) a part of a circle.

Proof:

- $\kappa = c$ so the signed curvature is either $\kappa_S(s) = c$ for all s or $\kappa_S(s) = -c$ for all s.
- The circle $\gamma_{c+}(s) = \frac{1}{c}(\cos(cs), \sin(cs))$ is unit speed; easy to check that it has $\kappa_S = c$.
- The circle $\gamma_{c-}(s) = \frac{1}{c}(\cos(cs), -\sin(cs))$ is unit speed; easy to check that it has $\kappa_S = -c$.
- By the theorem of the previous slides, the curve γ must be isometric to one of these two circles, with domain restricted to (a, b).

Lecture 5: Space curves

Cross-product: quick review

Recall $\mathbf{A} \times \mathbf{B}$:

- Orthonormal basis (ONB) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
- Let $A = (A_1, A_2, A_3) = A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + A_3\mathbf{e}_3$.
- Let $B = (B_1, B_2, B_3) = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3$.
- Then the cross-product $\mathbf{A} \times \mathbf{B}$ is the vector

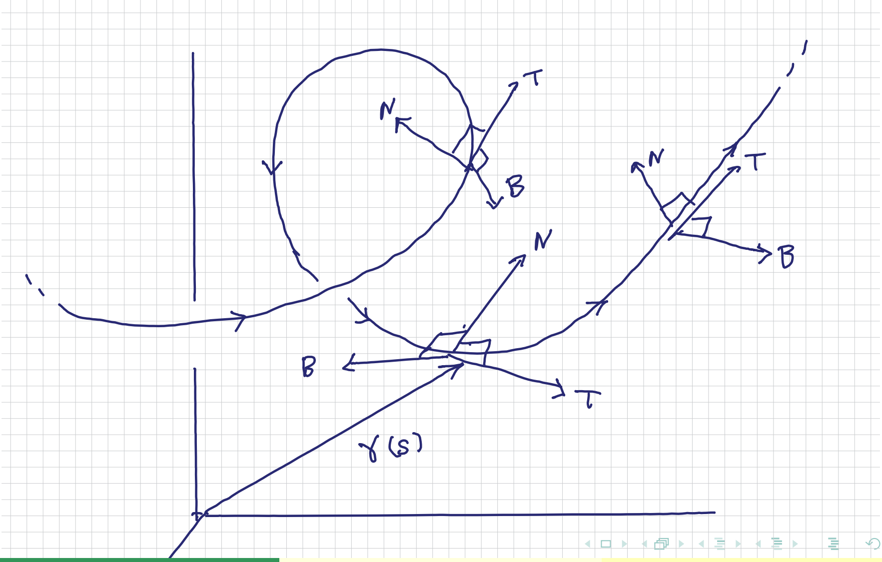
$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= \mathbf{e}_1 \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix} \\ &= (A_2B_3 - A_3B_2)\,\mathbf{e}_1 + (A_3B_1 - A_1B_3)\,\mathbf{e}_2 + (A_1B_2 - A_2B_1)\,\mathbf{e}_3\end{aligned}$$

- Recall: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, and so $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.
- $\mathbf{A} \times \mathbf{B} \perp \mathbf{A}$ and $\mathbf{A} \times \mathbf{B} \perp \mathbf{B}$.
- $\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{A}\|\,\|\mathbf{B}\|\,|\sin\theta|$, for θ then angle between $\mathbf{A}$ and $\mathbf{B}$.

Space curves

- Space curve $\gamma : I \to \mathbb{R}^3$.
- Assume γ to be unit speed: $\|\dot{\gamma}(s)\| = \sqrt{\dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2} = 1$.
- Then s is an arclength parameter.
- Unit tangent vector $\mathbf{T}(s) = \dot{\gamma}(s)$.
- Curvature $\kappa(s) = \|\ddot{\gamma}(s)\| = \left\|\dot{\mathbf{T}}\right\|$.
- *Principal unit normal* $\mathbf{N}(s) = \frac{1}{\kappa(s)}\ddot{\gamma}(s)$.
- $\mathbf{T} \perp \mathbf{N}$ if $\kappa \neq 0$. Proof:
    - $\mathbf{T} \cdot \mathbf{T} = 1 \implies 2\mathbf{T} \cdot \dot{\mathbf{T}} = 0$, so $\mathbf{T} \perp \dot{\mathbf{T}}$ (note that $\dot{\mathbf{T}} = \ddot{\gamma} \neq \mathbf{0}$ iff $\kappa \neq 0$).
    - Since $\dot{\mathbf{T}} = \kappa \mathbf{N}$ then $\mathbf{T} \perp \mathbf{N}$
- Define *binormal vector* $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.
- $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an ONB for $\mathbb{R}^3$ at each point of γ, called the *Frenet frame*.

Frenet frame



Frenet frame

For unit speed curves $\gamma(s)$ we have

- Unit tangent vector $\mathbf{T} = \dot{\gamma}(s)$.
- Principal unit normal vector $\mathbf{N} = \frac{1}{\kappa}\dot{\mathbf{T}} = \frac{1}{\kappa(s)}\ddot{\gamma}(s)$.
- Unit binormal vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Since Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an ONB, must have (up to a sign)

- $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
- $\mathbf{T} = \mathbf{N} \times \mathbf{B}$
- $\mathbf{N} = \mathbf{B} \times \mathbf{T}$

With these sign choices, the Frenet frame is a *right-handed* ONB.

Torsion

For a unit speed curve, we have $\dot{\mathbf{T}} = \kappa \mathbf{N}$. What about $\dot{\mathbf{B}}$ and $\dot{\mathbf{N}}$?

- Differentiate $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Get $\dot{\mathbf{B}} = \dot{\mathbf{T}} \times \mathbf{N} + \mathbf{T} \times \dot{\mathbf{N}}$.
- But $\dot{\mathbf{T}} \times \mathbf{N} = \kappa \mathbf{N} \times \mathbf{N} = \mathbf{0}$.
- Then $\dot{\mathbf{B}} = \mathbf{T} \times \dot{\mathbf{N}}$.
- Then $\dot{\mathbf{B}} \perp \mathbf{T}$.
- But also $\dot{\mathbf{B}} \perp \mathbf{B}$ (since $0 = \frac{d}{ds}(\mathbf{B} \cdot \mathbf{B}) = 2\mathbf{B} \cdot \dot{\mathbf{B}}$).
- Conclude that $\dot{\mathbf{B}}$ is parallel to $\mathbf{N}$ and write

$$\dot{\mathbf{B}}(s) =: -\tau(s)\mathbf{N}(s).$$

- This equation defines the *torsion* $\tau(s)$.

Formula for torsion

Unit speed curves $\gamma(s)$:

- Last slide: $\dot{\mathbf{B}} = \mathbf{T} \times \dot{\mathbf{N}}$ and $\dot{\mathbf{B}} =: -\tau \mathbf{N}$.
- $\implies -\tau \mathbf{N} = \mathbf{T} \times \dot{\mathbf{N}}$.
- Then $\tau = -\mathbf{N} \cdot \left(\mathbf{T} \times \dot{\mathbf{N}}\right) = \mathbf{N} \cdot \left(\dot{\mathbf{N}} \times \mathbf{T}\right)$.

Curves with arbitrary parametrization $\gamma(t)$:

- In above formula, replace $\mathbf{N}$ by $\mathbf{N}(s) = \frac{1}{\kappa(s)}\dot{\mathbf{T}}(s) = \frac{1}{\kappa(s)}\ddot{\gamma}(s)$.
- Use chain rule to write $\frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = \frac{d\gamma}{ds}\left\|\dot{\gamma}(t)\right\|$, and use formula for κ.
- Tedious calculation (text Prop 2.3.1) gives

$$\tau(t) = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\left\|\dot{\gamma} \times \ddot{\gamma}\right\|^2}.$$

Important point: When defining τ, needed to use $\mathbf{N} = \frac{1}{\kappa}\dot{\mathbf{T}}$.
$\implies$ *When $\kappa = 0$, cannot unambiguously define τ or even the Frenet frame.*

Meaning of torsion

- Curve γ with $\kappa \neq 0$ so $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and τ defined.
- Suppose $\tau(s) = 0$ for all s along γ.
- Then $\dot{\mathbf{B}} = -\tau \mathbf{N} \implies \dot{\mathbf{B}} = 0$, so $\mathbf{B}$ is a constant vector.
- Then $\frac{d}{ds}(\gamma \cdot \mathbf{B}) = \frac{d\gamma}{ds} \cdot \mathbf{B} = \mathbf{T} \cdot \mathbf{B} = 0$ since $\mathbf{T} \perp \mathbf{B}$.
- $\implies \gamma \cdot \mathbf{B} = d = const$ all along γ.
- But this is the equation of a plane
  with normal vector $\mathbf{B}$! To see this,
  if $\gamma = (x(s), y(s), z(s))$ and $\mathbf{B} = (a, b, c)$,
  then $\gamma \cdot \mathbf{B} = d$ becomes $ax + by + cz = d$.

Theorem

If a space curve $\gamma : I \to \mathbb{R}^3$ has $\tau(s) = 0$ for all $s \in I$, then it lies in a plane. The binormal $\mathbf{B}$ to γ is normal to the plane.

Note: If $\kappa = 0$ for all $s \in I$, γ is a line and lies in a plane; indeed many planes.

The converse

Theorem

If a space curve $\gamma : I \to \mathbb{R}^3$ with nonzero κ lies in a plane, it has $\tau(s) = 0$ for all $s \in I$.

Proof:

- Say plane has normal (a, b, c). Then γ obeys $(a, b, c) \cdot \gamma = d$.
- Differentiate. Get $(a, b, c) \cdot \dot{\gamma} = (a, b, c) \cdot \mathbf{T} = 0$, so $(a, b, c) \perp \mathbf{T}$.
- Differentiate again: $(a, b, c) \cdot \dot{\mathbf{T}} = (a, b, c) \cdot (\kappa \mathbf{N}) = 0$. Since $\kappa \neq 0$, then $(a, b, c) \perp \mathbf{N}$.
- Hence (a, b, c) is parallel to $\mathbf{B}$, and so $\mathbf{B}$ has constant direction. But $\mathbf{B}$ also has constant norm, so it's a constant vector; indeed, $\mathbf{B} = \pm(a, b, c)/\sqrt{a^2 + b^2 + c^2}$.
- Then $\dot{\mathbf{B}} = \mathbf{0}$. But $\dot{\mathbf{B}} = -\tau \mathbf{N}$. Therefore $\tau = 0$.

Note: By continuity, these theorems also hold if $\kappa = 0$ at isolated points along γ.

What is $\dot{\mathbf{N}}$?

For a unit speed curve $\gamma(s)$ with $\kappa \neq 0$:

- $\dot{\mathbf{T}} = \kappa \mathbf{N}$, and
- $\dot{\mathbf{B}} = -\tau \mathbf{N}$.
- Now $\mathbf{N} = \mathbf{B} \times \mathbf{T}$ so

$$\begin{aligned}\dot{\mathbf{N}} &= \dot{\mathbf{B}} \times \mathbf{T} + \mathbf{B} \times \dot{\mathbf{T}} \\ &= -\tau \mathbf{N} \times \mathbf{T} + \kappa \mathbf{B} \times \mathbf{N} = \tau \mathbf{T} \times \mathbf{N} - \kappa \mathbf{N} \times \mathbf{B} \\ &= \tau \mathbf{B} - \kappa \mathbf{T}.\end{aligned}$$

The Frenet-Serret equations are:

$$\begin{aligned}\dot{\mathbf{T}} &= \kappa \mathbf{N} \\ \dot{\mathbf{N}} &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \dot{\mathbf{B}} &= -\tau \mathbf{N}\end{aligned}$$

Matrix form:

$$\frac{d}{ds}\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

Notice the square matrix is skew symmetric (i.e., anti-symmetric).

Circles again

Theorem

If a unit speed space curve $\gamma : I \to \mathbb{R}^3$ has $\tau = 0$ and $\kappa = const \neq 0$ for all $t \in I$, then γ is part of a circle of radius $1/\kappa$.

Proof:

- $\tau = 0$ implies that γ lies in a plane Π.
- $\mathbf{B}$ is a constant vector field along γ normal to Π.
- $\dot{\mathbf{N}} = -\kappa \mathbf{T} + \tau \mathbf{B} = -\kappa \mathbf{T}$ so $\mathbf{T} + \frac{1}{\kappa}\dot{\mathbf{N}} = \mathbf{0}$.
- Since $\kappa = const$, can write last formula as $\frac{d}{ds}\left(\gamma + \frac{1}{\kappa}\mathbf{N}\right) = \mathbf{0}$.
- Integrate: $\gamma + \frac{1}{\kappa}\mathbf{N} = \mathbf{p}$ for $\mathbf{p} = (a, b, c) \in \Pi \subset \mathbb{R}^3$.
- $\implies \|\gamma(s) - \mathbf{p}\| = \frac{1}{\kappa} = const$.
- That's the equation of a sphere of radius $1/\kappa$ about centre $\mathbf{p}$.
- The curve is a great circle: intersection of the sphere with plane Π that contains the sphere's centre $\mathbf{p}$.

Fundamental theorem for space curves

Theorem

Let $\gamma : I \to \mathbb{R}^3$ and $\tilde{\gamma} : I \to \mathbb{R}^3$ be two unit speed curves with the same domain I, same curvature $\kappa(s)$, and same torsion $\tau(s)$ for all $s \in I$.
Then there is a direct isometry $M : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\tilde{\gamma}(s) = M(\gamma(s)) = (M \circ \gamma)(s) \text{ for all } s \in I.$$

Furthermore, if $k : I \to \mathbb{R}$ is a smooth positive function and if $t : I \to \mathbb{R}$ is a smooth function, then there is a unit speed curve $\gamma : I \to \mathbb{R}^3$ whose curvature is k and whose torsion is t.

Proof.

See text, pp 52–53. ☐

Something to think about: What might the fundamental theorem of curves look like in $\mathbb{R}^4$? in $\mathbb{R}^n$? in Minkowski spacetime (for those studying general relativity)?

Lecture 6: Isoperimetric inequality

Jordan curve theorem

Simple closed curves, also called *Jordan curves*, are closed plane curves that do not self-intersect.

Theorem (Jordan curve theorem)

Every simple closed curve separates $\mathbb{R}^2$ *into two disjoint regions,*
called the interior and exterior regions.
The interior region is bounded (contained within a circle).
The exterior region is unbounded.

Simple statement, surprisingly difficult to prove:
see graduate level algebraic topology texts for proof.

The isoperimetric inequality

Theorem

Let $\gamma : I \to \mathbb{R}^2$ *be a simple closed curve of length* $L(\gamma)$*, enclosing a region of area* $A(\gamma)$*. Then*

$$A(\gamma) \leq \frac{1}{4\pi} \left(L(\gamma)\right)^2 .$$

Equality holds iff γ *is a circle.*

This simple theorem has motivated a great many proofs and almost as many profound ideas. The most common proof uses

Theorem (Wirtinger's inequality)

Let $F : [0, \pi] \to \mathbb{R}$ *be a smooth function with* $F(0) = F(\pi) = 0$*. Then*

$$\int_0^\pi \left(\frac{dF}{dt}\right)^2 dt \geq \int_0^\pi \left(F(t)\right)^2 dt,$$

and equality holds iff $F(t) = C \sin t$*,* $C = const$*.*

Proof of isoperimetric inequality

- Unit speed closed curve γ, arclength L, positioned so that $\gamma(0) = \mathbf{0}$.
- Reparametrize by $t = \frac{\pi s}{L}$. Then $t \in [0, \pi]$, speed is $\|\dot{\gamma}(t)\| = \frac{L}{\pi} = const$.
- Polar coordinates: $\gamma(t) = (r(t), \theta(t))$. Then

$$L^2 = \pi^2 \|\dot{\gamma}(t)\|^2 = \pi \int_0^\pi \|\dot{\gamma}(t)\|^2 \, dt = \pi \int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2\right) dt. \tag{1}$$

- From Calculus, area enclosed by a polar curve is

$$A = \frac{1}{2} \int_0^\pi (x\dot{y} - \dot{x}y) \, dt = \frac{1}{2} \int_0^\pi r^2(t)\dot{\theta}(t) dt. \tag{2}$$

- Combine (1) and (2):

$$\frac{L^2}{4\pi} - A = \frac{1}{4} \int_0^\pi \left(\dot{r}^2 + r^2\dot{\theta}^2 - 2r^2\dot{\theta}\right) dt = \frac{1}{4} \int_0^\pi \left[\dot{r}^2 + r^2\left(\dot{\theta}^2 - 2\dot{\theta}\right)\right] dt.$$

Isoperimetric inequality continued

- Complete the square:

$$\begin{aligned}\frac{L^2}{4\pi} - A &= \frac{1}{4}\int\limits_0^\pi \left[\dot{r}^2 - r^2 + r^2\left(\dot{\theta}-1\right)^2\right] dt \\ &\geq \frac{1}{4}\int\limits_0^\pi \left[\dot{r}^2 - r^2\right] dt \\ &\geq 0\end{aligned} \tag{3}$$

  by Wirtinger's inequality, which we recall says that $\int\limits_0^\pi \dot{r}^2 dt \geq \int\limits_0^\pi r^2 dt$ for any smooth function $r(t)$ such that $r(0) = r(\pi) = 0$.

- This proves the inequality.

Case of equality

We still have to show that $\frac{L^2}{4\pi} = A$ iff γ is a circle.

- If γ is a circle, then $L = 2\pi r$ so $\frac{L^2}{4\pi} = \pi r^2$.
- But if γ is a circle, then $A = \pi r^2$. Hence $\frac{L^2}{4\pi} = A$.
- Must prove converse: that if $\frac{L^2}{4\pi} = A$ then γ is a circle.
- Use $\frac{L^2}{4\pi} - A = 0$ in first line of (3):

$$\begin{aligned} 0 = \frac{L^2}{4\pi} - A &= \frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2 + r^2\left(\dot{\theta} - 1\right)^2\right] dt \\ &= \frac{1}{4}\int_0^\pi \left[\dot{r}^2 - r^2\right] dt + \frac{1}{4}\int_0^\pi r^2\left(\dot{\theta} - 1\right)^2 dt \end{aligned}$$

Equality case continued

- Last slide: $0 = \frac{1}{4}\int\limits_0^\pi \left[\dot{r}^2 - r^2\right] dt + \frac{1}{4}\int\limits_0^\pi r^2\left(\dot{\theta} - 1\right)^2 dt$.
- By Wirtinger, first integral on right is nonnegative. Second integral on right is obviously nonnegative. Thus, each integral must vanish:

$$\int\limits_0^\pi \left[\dot{r}^2 - r^2\right] dt = 0 \quad \text{and} \quad \int\limits_0^\pi r^2\left(\dot{\theta} - 1\right)^2 dt = 0.$$

- But $\int\limits_0^\pi r^2\left(\dot{\theta} - 1\right)^2 dt = 0 \implies \dot{\theta} = 1 \implies \theta = t + \theta_0$ for $\theta_0 = const$. Simplify: Rotate axes to get $\theta_0 = 0$, then $\theta = t$.
- And $\frac{1}{4}\int\limits_0^\pi \left[\dot{r}^2 - r^2\right] dt = 0 \implies r = C\sin t$ by the equality case of Wirtinger.
- So $r = C\sin\theta$, which is polar equation of *circle* that passes through the origin. (Exercise: Obtain the Cartesian form $x^2 + \left(y - \frac{C}{2}\right)^2 = \frac{C^2}{4}$.)

Addendum: Sketch of proof of Wirtinger's inequality

Set-up:

- Define $G(t) = F(t)/\sin t$, $t \in (0, \pi)$.
- $\lim_{t\to 0^+} G(t) = \lim_{t\to 0^+} \frac{F(t)}{\sin t} = \lim_{t\to 0^+} \frac{F'(t)}{\cos t} = \lim_{t\to 0^+} F'(t)$. Exists because F is smooth. Likewise, $\lim_{t\to\pi^-} G(t)$ exists. So define $G(0)$, $G(\pi)$ by continuity (i.e., $G(0) := \lim_{t\to 0^+} G(t)$).
- Then $G : [0, \pi] \to \mathbb{R}$ is smooth.
- Then $F(t) = G(t)\sin t$, so $\dot{F}(t) = \dot{G}(t)\sin t + G(t)\cos t$.
- Use this an integration by parts (details: text p 61) to compute

$$\int\limits_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = \int\limits_0^\pi \dot{G}^2(t)\sin^2 t dt \geq 0.$$

- This proves the inequality.

Addendum: Sketch of equality case

- Last slide: $\int\limits_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = \int\limits_0^\pi \dot{G}^2(t) \sin^2 t dt \geq 0$.
- From this, if $\int\limits_0^\pi \left(\dot{F}^2(t) - F^2(t)\right) dt = 0$, then necessarily $\int\limits_0^\pi \dot{G}^2(t) \sin^2 t dt = 0$.
- Because the integrand is nonnegative, the integral is zero only if $\dot{G}(t) \sin t = 0$ for all $t \in [0, \pi]$.
- Then $\dot{G}(t) = 0$, so $G(t) = C = const$.
- Since $G(t) = F(t)/\sin t$, we have $F(t) = C \sin t$, as required.

Lecture 7: What is a surface?

Review some basic concepts

Definition

An *open set* in $\mathbb{R}^n$ is a set S that contains a neighbourhood of each of its points. That is, if $p \in S$, then there is an $\epsilon > 0$ such that $q \in S$ whenever $\|p - q\| < \epsilon$.

- The *ball* of radius $a > 0$, $\left\{ p \in \mathbb{R}^2 \;\middle|\; \|p\| < a \right\}$, is open.
- The *closed ball* of radius $a > 0$, $\left\{ p \in \mathbb{R}^2 \;\middle|\; \|p\| \leq a \right\}$, is *not* open.

Definition

Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$. The function $f : X \to Y$
is *continuous* at x_0 if, given that $f(x_0) = y_0$,
then points near x_0 are mapped to points near y_0.
That is, f is continuous at x_0 if, for any $\epsilon > 0$,
we can make $|f(x) - f(x_0)| < \epsilon$
whenever $|x - x_0| < \delta$ for some $\delta > 0$.

Homeomorphism

Definition (Equivalent definition of continuity)

$f : X \to \mathbb{R}^n$ with $X \subset \mathbb{R}^m$ is continuous if and only if for every open set $V \subset \mathbb{R}^n$ there is an open set $U \subset \mathbb{R}^m$ such that $U \cap X = \{x \in X | f(x) \in V\}$.

Definition

If $f : X \to Y$ is continuous and bijective (injective and surjective; in other words, one-to-one and onto) and if $f^{-1} : Y \to X$ is continuous, then f is called a *homeomorphism*, and we say that X and Y are *homeomorphic*.

Definition of a surface

Definition

A subset $S \subset \mathbb{R}^3$ is a *surface* if for every point $p \in S$ there are open sets $U \subset \mathbb{R}^2$ and $W \subset \mathbb{R}^3$ with $p \in W$ such that $S \cap W$ is homeomorphic to U.

- A homeomorphism $X : U \to S \cap W$ is called a *surface patch* or a *parametrization* of $S \cap W$.
- For $(u, v) \in U \subset \mathbb{R}^2$, $X(u, v)$ is a *parametrized surface*.
- A collection of surface patches whose union covers S is an *atlas* for S.
- Notation: Text uses $\sigma : U \to S \cap W$ where I used $X : U \to S \cap W$.

Example: Planes

- Every plane Π in $\mathbb{R}^3$ is a surface with an atlas consisting of just one patch.
- Let $(u, v) \in \mathbb{R}^2$.
- Let $\mathbf{p} \perp \mathbf{q}$ be vectors tangent to Π.
- Let $\mathbf{a}$ be a fixed point in Π.
- Then $X(u, v) = \mathbf{x} = \mathbf{a} + u\mathbf{p} + v\mathbf{q}$.
- Inverse mapping: $X^{-1}(\mathbf{x}) = (u, v) = ((\mathbf{b} - \mathbf{a}) \cdot \mathbf{p}, (\mathbf{b} - \mathbf{a}) \cdot \mathbf{q})$ since $\mathbf{p} \perp \mathbf{q}$.

Smooth surfaces

Definition

A function $f : U \to \mathbb{R}^n$ from an open set $U \subset \mathbb{R}^m$ is *smooth* if each component f_i of f is continuous in each argument and has continuous partial derivatives at all orders at every $\mathbf{u} = (u_1, \ldots, u_m) \in U$. If f is smooth, we sometimes write $f \in C^\infty(\mathbb{R}^n)$ or simply $f \in C^\infty$.

- A surface patch $X : U \to \mathbb{R}^3$ may or may not be smooth.
- Example: the single-napped cone $X(u, v) = (u, v, \sqrt{u^2 + v^2})$ has no smooth patches containing origin $(u, v) = (0, 0)$.

Regular patch

Definition

A surface patch $X : U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$, is *regular* if it is smooth and the vectors

$$X_u = \frac{\partial X}{\partial u} = \left(\frac{\partial X_1}{\partial u}, \frac{\partial X_2}{\partial u}, \frac{\partial X_3}{\partial u} \right)$$
$$X_v = \frac{\partial X}{\partial v} = \left(\frac{\partial X_1}{\partial v}, \frac{\partial X_2}{\partial v}, \frac{\partial X_3}{\partial v} \right)$$

are *linearly independent*; equivalently, if $X_u \times X_v \neq \mathbf{0}$ for all $(u, v) \in U$.

When this condition holds, the set $\{X_u, X_v\}$ is a *basis set* for the tangent plane to the surface at the point $X(u, v)$.

Allowable patches and atlases

Definition

If $X : U \to \mathbb{R}^3$ is a regular surface patch and if X is a homeomorphism from U to an open subset of S then $X : U \to \mathbb{R}^3$ is an *allowable* surface patch.

Definition

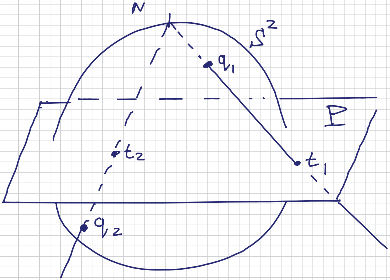
A *smooth surface* is a surface S such that, for each $p \in S$, there is an allowable surface patch $X : U \to \mathbb{R}^3$ with $p \in X(U)$.

Definition

A collection of allowable surface patches for a surface S such that each $p \in S$ belongs to at least one patch is an *atlas* for S. A *maximal atlas* for a smooth surface S is one that contains every allowable surface patch for S.

Example: Stereographic projection

- Project $\mathbb{S}^2 \to \mathbb{R}^2$
- $\mathbb{S}^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\}$.
- $P = \{(x, y, z) | z = 0\}$
- Draw line from north pole N, meets $q \in \mathbb{S}^2$ and $t \in P$.
- This patch, call it σ_N, maps t to q,
- Patch covers every point of $\mathbb{S}^2$ except N.
- A similar patch σ_S covers every point of $\mathbb{S}^2$ except the south pole S.



$\sigma_N(t_1) = q_1$, $\sigma_N(t_2) = q_2$

Patches for stereographic projection

- Projection from $N = (0,0,1)$ gives the patch

$$\begin{aligned}\sigma_N(u,v) &= (x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3 \text{ where } (u,v) \in P \subset \mathbb{R}^2 \\ &= \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)\end{aligned}$$

- Projection from $S = (0,0,-1)$ gives the patch

$$\begin{aligned}\sigma_S(\tilde{u},\tilde{v}) &= (x,y,z) \in \mathbb{S}^2 \subset \mathbb{R}^3 \text{ where } (u,v) \in P \subset \mathbb{R}^2 \\ &= \left(\frac{2\tilde{u}}{\tilde{u}^2+\tilde{v}^2+1}, \frac{2\tilde{v}}{\tilde{u}^2+\tilde{v}^2+1}, \frac{1-\tilde{u}^2-\tilde{v}^2}{\tilde{u}^2+\tilde{v}^2+1} \right)\end{aligned}$$

- Together, both patches cover $\mathbb{S}^2$. They form an atlas for $\mathbb{S}^2$.
- The patches overlap: every point of $\mathbb{S}^2$ except $N = (0,0,1)$ and $S = (0,0,-1)$ lies in both patches.

Transition maps

Definition

If two coordinate patches $X : U \to \mathbb{R}^3$ and $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ overlap on a region $V \subset \mathbb{R}^3$, we can define *transition maps*

$$\Phi := X^{-1} \circ \tilde{X} : \tilde{U} \to U$$
$$\tilde{X}(\tilde{u}, \tilde{v}) = X(u, v) = X(\Phi(\tilde{u}, \tilde{v})) = (X \circ \Phi)(\tilde{u}, \tilde{v}).$$

Theorem

The transition maps of a smooth surface are smooth maps.

- The proof of this theorem is in Chapter 5 of the text.
- Transition maps are sometimes called *coordinate transformations*.

Jacobian determinants

- Assume $X : U \to \mathbb{R}^3$ is a regular surface patch, $U \in \mathbb{R}^2$ is open, $\Phi : \tilde{U} \to U$ is a smooth bijection.
- Then $\tilde{X} = \sigma \circ \Phi$ is smooth. We have $\tilde{X}(\tilde{u}, \tilde{v}) = X(u, v) = X \circ \Phi(\tilde{u}, \tilde{v})$.
- Chain rule: $\tilde{X}_{\tilde{u}} = \frac{\partial \tilde{X}}{\partial \tilde{u}} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \tilde{u}} = \frac{\partial u}{\partial \tilde{u}} X_u + \frac{\partial v}{\partial \tilde{u}} X_v$.
- Likewise: $\tilde{X}_{\tilde{v}} = \frac{\partial \tilde{X}}{\partial \tilde{v}} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \tilde{v}} = \frac{\partial u}{\partial \tilde{v}} X_u + \frac{\partial v}{\partial \tilde{v}} X_v$.
- Matrix form: $\left[\begin{array}{c} \tilde{X}_{\tilde{u}} \\ \tilde{X}_{\tilde{v}} \end{array}\right] = \left[\begin{array}{cc} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{array}\right] \left[\begin{array}{c} X_u \\ X_v \end{array}\right] = [J(\Phi)] \left[\begin{array}{c} X_u \\ X_v \end{array}\right]$ (Note: $\left[\begin{array}{c} X_u \\ X_v \end{array}\right]$ and $\left[\begin{array}{c} \tilde{X}_{\tilde{u}} \\ \tilde{X}_{\tilde{v}} \end{array}\right]$ are 2×3 matrices, not column vectors.)
- The *Jacobian matrix* is $[J(\Phi)] = \left[\begin{array}{cc} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{array}\right]$. Its determinant is the *Jacobian determinant* or simply the *Jacobian* of Φ.

Jacobian determinants continued

- From last slide:

$$\tilde{X}_{\tilde{u}} = \frac{\partial u}{\partial \tilde{u}} X_u + \frac{\partial v}{\partial \tilde{u}} X_v$$
$$\tilde{X}_{\tilde{v}} = \frac{\partial u}{\partial \tilde{v}} X_u + \frac{\partial v}{\partial \tilde{v}} X_v$$

- Then

$$\begin{aligned}\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} &= \left(\frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{v}} - \frac{\partial u}{\partial \tilde{v}} \frac{\partial v}{\partial \tilde{u}} \right) X_u \times X_v \\ &= \left(\det J(\Phi) \right) X_u \times X_v .\end{aligned}$$

- The formula

$$\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = \left(\det J(\Phi) \right) X_u \times X_v$$

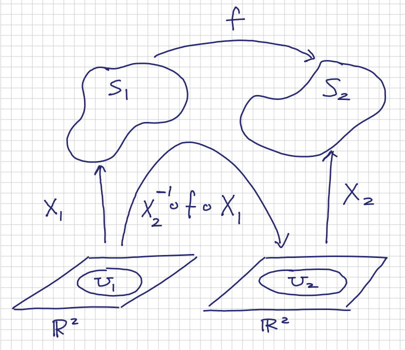
will be important later.

Properties of Jacobians

- Three overlapping patches X, $\tilde{X} = X \circ \Phi$, and $\hat{X} = \tilde{X} \circ \tilde{\Phi}$, so that
$$\hat{X} = \tilde{X} \circ \tilde{\Phi} = (X \circ \Phi) \circ \tilde{\Phi} = X \circ (\Phi \circ \tilde{\Phi}).$$
- Then $[J(\Phi \circ \tilde{\Phi})] = [J(\Phi)][J(\tilde{\Phi})]$ (proof: use chain rule).
- Special case: $\tilde{\Phi} = \Phi^{-1}$. Then $[J(\Phi \circ \Phi^{-1})] = [J(\Phi)][J(\Phi^{-1})]$.
- But $\Phi \circ \Phi^{-1} = \text{id} = \text{identity map id}(u, v) = (u, v)$, so $[J(\Phi \circ \Phi^{-1})] = [J(\text{id})] = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies [J(\Phi)][J(\Phi^{-1})] = \mathbb{I}$.
- Conclude that $[J(\Phi)]$ is invertible, so $\det[J(\Phi)] \neq 0$.
- Indeed, $[J(\Phi)]^{-1} = [J(\Phi^{-1})]$.
- If X is regular, then $X_u \times X_v \neq \mathbf{0}$. Since $\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = (\det J(\Phi))\, X_u \times X_v$ and now $\det[J(\Phi)] \neq 0$, then $\tilde{X}$ is regular too.
- Theorem: Let $U, \tilde{U} \subset \mathbb{R}^2$ be open and let $X : U \to \mathbb{R}^3$ be a regular surface patch. Let $\Phi : U \to \tilde{U}$ be a smooth bijection with smooth inverse. Then $\tilde{X} = X \circ \Phi : \tilde{U} \to \mathbb{R}^3$ is a regular surface patch.

Smooth maps between smooth surfaces

- Smooth surfaces S_1, S_2.
- Patch $X_1 : U_1 \to \mathbb{R}^3$ covers S_1
- Patch $X_2 : U_2 \to \mathbb{R}^3$ covers S_2.
- Map $f := S_1 \to S_2$ is *smooth* if the map $X_2^{-1} \circ f \circ X_1$ from U_1 to U_2 is smooth.
- Well-defined: If f is smooth using patches X_1, X_2, it is smooth using any other smooth patches.



Diffeomorphisms

Definition

If $f : S_1 \to S_2$ is smooth and bijective and $f^{-1} : S_2 \to S_1$ is smooth, then f is a *diffeomorphism* and we say that S_1 and S_2 are *diffeomorphic*.

Theorem

If $f : S_1 \to S_2$ is a diffeomorphism and $X_1 : U \to S_1$ is an allowable surface patch for S_1, then $X_2 := f \circ X_1 : U \to S_2$ is an allowable surface patch for S_2.

Proof: text p 83.

Definition

If $f : S_1 \to S_2$ is smooth, say that about
each $p \in S_1$ there's an open set $O_p \ni p$ such that
$f(O_p)$ is open in S_2, and say that
$f|_{O_p} : O_p \to f(O_p)$ is a diffeomorphism.
Then we say that f is a *local diffeomorphism*.

Lecture 8: Tangents, normals, orientations

Tangents

- Say $\gamma : I \to \mathbb{R}^3$ is a smooth space curve, with image in surface S.
- Then tangent vector $\dot{\gamma}(t_0)$ to γ at $p = \gamma(t_0)$ is tangent to S at p.
- The set of all tangents vectors to to curves in S through p is the *tangent space* (or *tangent plane*) T_pS to S at p.

Theorem

Let $X : U \to \mathbb{R}^3$ be a regular surface patch for surface S. Let $p \in S$. Let $(u, v) \in U$. Then T_pS is the subspace of $\mathbb{R}^3$ spanned by the vectors $\{X_u, X_v\}$.

Proof

- Curve $(u(t), v(t)) \in U \subset \mathbb{R}^2$.
- Use X to lift to curve $\gamma(t) = X(u(t), v(t))$ in S.
- Let $p = \gamma(t_0) = X(u_0, v_0)$.
- $\dot{\gamma}(t) = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt} = X_u \dot{u} + X_v \dot{v}$.
- Hence tangent vector $\dot{\gamma}(t_0)$ at p belongs to $\text{Span}\{X_u(t_0), X_v(t_0)\}$.
- Conversely, any vector $\mathbf{w} \in \text{Span}\{X_u(u_0, v_0), X_v(u_0, v_0)\}$ can be written as $\mathbf{w} = aX_u(t_0) + bX_v(t_0)$.
- Define curve $\gamma(t) = X(u_0 + a(t - t_0), v_0 + b(t - t_0))$.
- Then $\gamma(t_0) = X(u_0, v_0)$ and $\dot{\gamma}(t_0) = aX_u(u_0, v_0) + bX_v(u_0, v_0) = \mathbf{w}$.
- Hence any $\mathbf{w} \in \text{Span}\{X_u(u_0, v_0), X_v(u_0, v_0)\}$ is tangent to a curve in S through p, and so is in T_pS.

Dimension and basis

- Last theorem: for regular patch $X : U \to \mathbb{R}^3 : (u, v) \mapsto p$, then $\{X_u, X_v\}$ spans TpS.
- X is a regular patch so $X_u \times X_v \neq \mathbf{0}$.
- Then $\{X_u, X_v\}$ is a linearly independent set.
- A linearly independent spanning set is a basis set.
- $\{X_u, X_v\}$ is a basis for T_pS.
- Then T_pS is 2-dimensional.

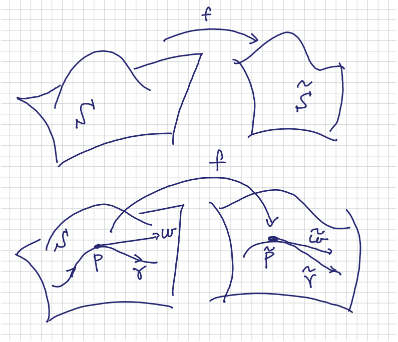
Parameter curves

- Let $u_0, v_0 \in \mathbb{R}$ be constants and $(u_0, v_0) \in U$.
- Let $X : U \to \mathbb{R}^3$ be a surface patch for surface S.
- The map $u \mapsto X(u, v_0)$ is a curve (i.e., $\gamma(u) = (u, v_0)$).
- The map $v \mapsto X(u_0, v)$ is a curve.

- These maps are called *parameter curves* or *coordinate curves*.
- Their tangents are $X_u(u, v_0)$ and $X_v(u_0, v)$ respectively.

Pushforwards

- $f : S \to \tilde{S}$ is a smooth map between surfaces (or possibly from S to itself).
- $p \in S$, $\mathbf{w} \in T_pS$, where $\mathbf{w} = \dot{\gamma}(t_0)$ is tangent to some curve γ at $\gamma(t_0) = p$.
- Let $\tilde{\gamma}$ be the curve $f \circ \gamma$ through $\tilde{p} = f(p)$, and let $\tilde{w} = \dot{\tilde{\gamma}}(t_0)$ be tangent to $\tilde{\gamma}$ at $f(p)$.
- We call $\tilde{\mathbf{w}}$ the *pushforward* of $\mathbf{w}$.



Derivative of a diffeomorphism

- Recall linear approximation in Calculus: $\Delta y = f'(x_0)\Delta x$.
- Derivatives convert "tangent vectors" $\frac{\Delta x}{\Delta t}$ along curves $x(t)$ to "tangent vectors" $\frac{\Delta y}{\Delta t}$ along curves $y(t)$ where $y = f(x)$.

Definition (Derivative of a diffeomorphism)

- The *derivative* of f at $p \in S$ is the linear map $D_p f : T_p S \to T_{f(p)}\tilde{S}$ defined such that $D_p f(\mathbf{w}) = \tilde{\mathbf{w}}$ for any $\mathbf{w} \in T_p S$.
- In a patch $X : U \to \mathbb{R}^3$ with $p = X(u_0, v_0)$, the *components* of $D_p f$ are the partial derivatives of $f \circ X$ along the parameter curves:

$$(D_p f)(X_u) = \frac{d}{du}\Big|_{u=u_0} f(X(u, v_0)) \ , \ (D_p f)(X_v) = \frac{d}{dv}\Big|_{v=v_0} f(X(u_0, v)) \ .$$

- Infinitely many curves through p with tangent $\mathbf{w}$.
- Definition of derivative does not depend on which such curve we use: text p 87.

Normals and orientability

- Every plane P in $\mathbb{R}^3$ has
  - infinitely many normals (if $\mathbf{N}$ is normal to P, so is $k\mathbf{N}$ for any $k \neq 0$), but
  - two unit normals $\pm\mathbf{N}$, where $\mathbf{N}$ is normal to P and $\|\mathbf{N}\| = 1$.
- If $X : U \to \mathbb{R}^3$ is a regular patch for surface S and if $p = X(u_0, v_0) \in S$, then $\{X_u(u_0, v_0), X_v(u_0, v_0)\}$ is a basis for TpS. This gives a *unique* choice of normal:

$$\mathbf{N}_X := \frac{X_u \times X_v}{\|X_u \times X_v\|} \text{ at } p = X(u_0, v_0).$$

  This choice is called the *standard unit normal* for the patch X.

Orientations

- If $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ is another regular patch then $\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = (\det J(\Phi)) X_u \times X_v$, J=Jacobian, $\Phi : \tilde{U} \to U$ is the transition map.
- Then $\tilde{\mathbf{N}}_{\tilde{X}} = \frac{\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}}}{\left\| \tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} \right\|} = \frac{\det J}{|\det J|} \frac{X_u \times X_v}{\| X_u \times X_v \|} = \frac{\det J}{|\det J|} \mathbf{N}_X = \begin{cases} \mathbf{N}_X, & \det J > 0, \\ -\mathbf{N}_X, & \det J < 0. \end{cases}$

Definition

A surface S is *orientable* if there exists an atlas $\mathcal{A}$ for S such that, if Φ is the transition map between any two surface patches in $\mathcal{A}$, then $\det(J(\Phi)) > 0$ wherever Φ is defined.

Final points

Theorem

If S is an orientable surface with an atlas $\mathcal{A}$ as in the definition, then there is a smooth choice of unit normal at every point of S.

Proof: Take the standard unit normal in each patch in $\mathcal{A}$. By the above calculation, $\tilde{\mathbf{N}}_{\tilde{X}} = \mathbf{N}_X$ whenever patches overlap.

Definition (Orientation)

Such a choice of smooth unit normal at every point of S is called an *orientation* for S, and then S is said to be *oriented*.

To state the obvious, any oriented surface is orientable.

Examples (see handwritten PDF notes):

- The Möbius band (not orientable).
- The 2-dimensional torus (orientable).

In these examples:

1. standard 2-torus in $\mathbb{R}^3$ and
2. the Möbius band,

I denote coordinate patches using the textbook's notation

$$\sigma : U \to \mathbb{R}^3 \quad , \quad U \subset \mathbb{R}^2$$

instead of the notation

$$X : U \to \mathbb{R}^3$$

that I usually use in my PDF slides.

CR 4 plus: Closed surfaces

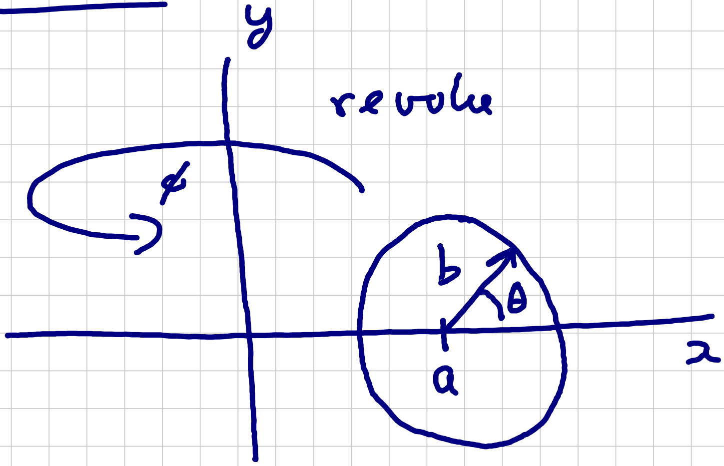
The standard torus:

$x(\theta, \phi) = (a + b\cos\theta)\cos\phi$

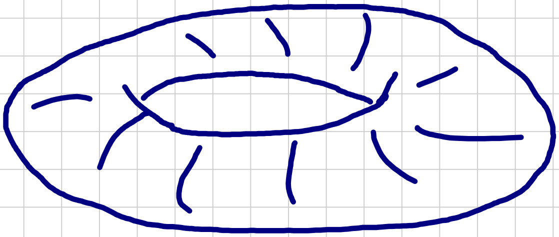
$y(\theta, \phi) = (a + b\cos\theta)\sin\phi$

$z(\theta, \phi) = b\sin\theta$

$a \geq b > 0.$



Here θ, ϕ are the parameters that we usually call u, v.



Often textbooks write that

$0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq 2\pi$, or $(\theta, \phi) \in [0, 2\pi] \times [0, 2\pi]$.

This is correct, but this is a <u>closed</u> domain.

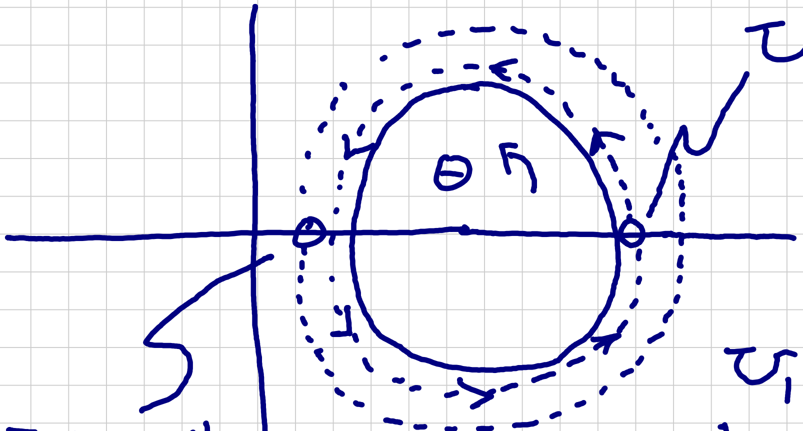
Our definition of surfaces requires surface patches to be defined with <u>open</u> domains.

$$\left.\begin{array}{l}
1.\ U_1 = \{(\theta, \phi) \mid 0 < \theta < 2\pi,\ 0 < \phi < 2\pi\} \\
2.\ U_2 = \{(\theta, \phi) \mid 0 < \theta < 2\pi,\ -\pi < \phi < \pi\} \\
3.\ U_3 = \{(\theta, \phi) \mid -\pi < \theta < \pi,\ 0 < \phi < 2\pi\} \\
4.\ U_4 = \{(\theta, \phi) \mid -\pi < \theta < \pi,\ -\pi < \phi < 2\pi\}
\end{array}\right\}
\begin{array}{l}\text{4 open domains} \\ \text{that, together,} \\ \text{cover the} \\ \text{standard torus.}\end{array}$$

On each domain, use the same <u>formula</u> for the homeomorphism:

$$\sigma(\theta, \phi) = \left((a + b\cos\theta)\cos\phi,\ (a + b\cos\theta)\sin\phi,\ b\sin\theta\right)$$

Compare two such domains, say U_1 and U_3.



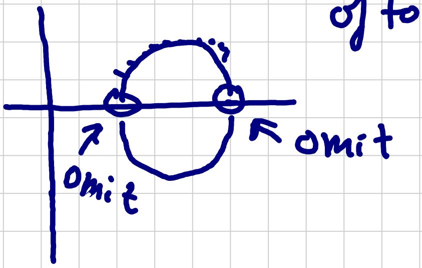
U_1 omits only $\theta = 0, 2\pi$

U_3 omits only $\theta = -\pi, \pi$.

U_1 and U_2 overlap in two disjoint regions.

$U_1 \cap U_2 =$ "top of torus" $\cup$ "bottom of torus"

$=$



That is, $U_1 \cap U_2 = \{(\theta, \phi) \mid 0 < \theta < \pi,\ 0 < \phi < 2\pi\}$

$$\cup \{(\theta, \phi) \mid \underbrace{\pi < \theta < 2\pi}_{\text{could also write } -\pi < \theta < \pi \text{ here.}},\ 0 < \phi < 2\pi\}$$

On these two regions, we have

$(\theta_3, \phi_3) = (\theta_1, \phi_1)$ on the "top region" and

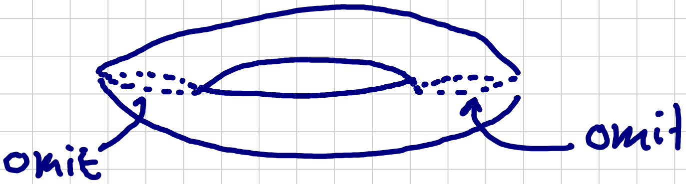
$(\theta_3, \phi_3) = (\theta_1 - 2\pi, \phi_1)$ on the bottom, where

$(\theta_1, \phi_1) \in U_1$, $(\theta_3, \phi_3) \in U_3$. If $\Phi(\theta_3, \phi_3) = (\theta_1, \phi_1)$

then $J(\Phi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on both the top and the bottom regions.

Similarly,

$$U_1 \cap U_2 = \{(\theta, \phi) \mid 0 < \theta < 2\pi,\ 0 < \phi < \pi\}$$
$$\cup \{(\theta, \phi) \mid 0 < \theta < 2\pi,\ \pi < \phi < 2\pi\}$$



On every overlap region $U_i \cap U_j$, we get transition functions Φ_{ij} with

$$J(\Phi_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

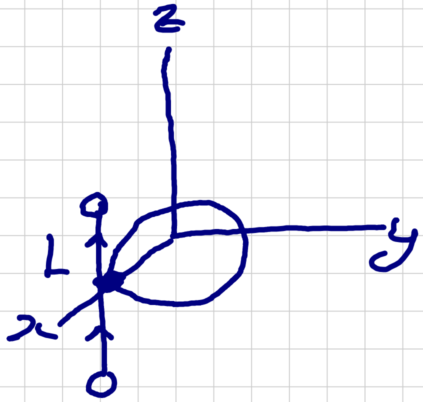
and so $\det J(\Phi_{ij}) = 1$.

$\Longrightarrow$ The standard torus is <u>orientable</u>.

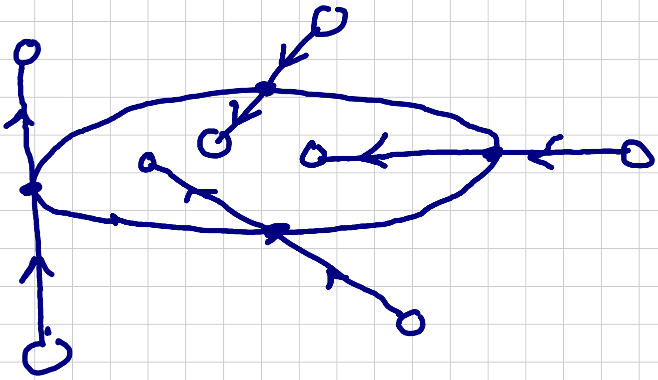
Example: Möbius band

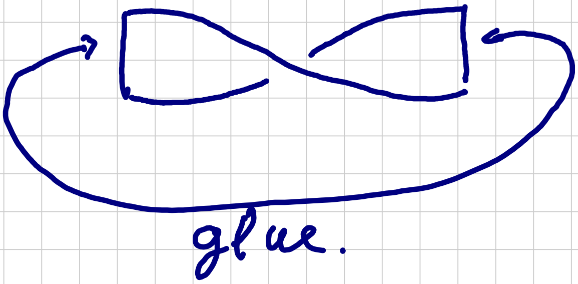
unit circle in $z=0$ plane
$x^2+y^2=1$

At $(1,0,0)$, attach the vertical line segment L
$\{(1,0,t) \mid t \in (-\frac{1}{2},\frac{1}{2})\}$
to the circle.



Now move the segment around the circle, always attached to the circle at its midpoint, so that L rotates by angle π (in the plane containing the point of attachment and the z-axis) as the point of attachment traverses the circle once.





Coordinate patches for the Möbius band

$$\sigma_i : U_i \to \mathbb{R}^3, \quad i = 1, 2$$

Both patches use the same formula

$$\sigma_1(t, \theta) = \sigma_2(t, \theta) =: \sigma(t, \theta)$$

$$= \left(\left(1 - t\sin\frac{\theta}{2}\right)\cos\theta,\ \left(1 - t\sin\frac{\theta}{2}\right)\sin\theta,\ t\cos\frac{\theta}{2}\right)$$

but different domains

$$U_1 = \left\{ (t, \theta) \in \mathbb{R}^2 \;\middle|\; -\tfrac{1}{2} < t < \tfrac{1}{2},\ 0 < \theta < 2\pi \right\}$$

$$U_2 = \left\{ (t, \theta) \in \mathbb{R}^2 \;\middle|\; -\tfrac{1}{2} < t < \tfrac{1}{2},\ -\pi < \theta < \pi \right\}$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial \sigma}{\partial t} &= \left(-\sin\frac{\theta}{2}\cos\theta,\ -\sin\frac{\theta}{2}\sin\theta,\ \cos\frac{\theta}{2}\right) \\ \left.\frac{\partial \sigma}{\partial \theta}\right|_{t=0} &= (-\sin\theta,\ \cos\theta,\ 0) \end{aligned}\right\} \begin{array}{l} \text{Median} \\ \text{circle } t = 0 \\ (x^2 + y^2 = 1, z = 0) \end{array}$$

$$\left.\frac{\partial \sigma}{\partial t}\right|_{t=0} \times \left.\frac{\partial \sigma}{\partial \theta}\right|_{t=0} = \begin{vmatrix} e_1 & e_2 & e_3 \\ -\sin\frac{\theta}{2}\cos\theta & -\sin\frac{\theta}{2}\sin\theta & \cos\frac{\theta}{2} \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$$

$$= \left(-\cos\theta\cos\frac{\theta}{2},\ -\sin\theta\cos\frac{\theta}{2},\ -\sin\frac{\theta}{2}\right)$$

This is a unit vector. Then

$$N = \left.\frac{\partial \sigma}{\partial t}\right|_{t=0} \times \left.\frac{\partial \sigma}{\partial \theta}\right|_{t=0} \quad \text{(divided by its magnitude, which is 1)}$$

is normal to the Möbius band along its median circle.

$$\left.\begin{array}{l}\text{Using patch } \sigma_2 \text{, we have} \\ \quad N = (-1,0,0) \text{ at } \theta = 0 \\ \text{Using patch } \sigma_1 \text{, we have} \\ \quad N \to (1,0,0) \text{ as } \theta \to 2\pi^-. \end{array}\right\} \Rightarrow \begin{array}{l}\text{The Möbius band} \\ \text{is not orientable.}\end{array}$$

N.B. Why were two patches necessary?

Answer: First, each point must belong to some patch. Now try the following alternatives:

(i) $U = \{(t,\theta) \mid -\frac{1}{2} < t < \frac{1}{2},\ 0 \leq \theta \leq 2\pi\}$

This is not open in $\mathbb{R}^2$.

(ii) $U = \{(t,\theta) \mid -\frac{1}{2} < t < \frac{1}{2},\ -\varepsilon < \theta < 2\pi + \varepsilon,\ \varepsilon > 0\}$

This is open in $\mathbb{R}^2$, but then the standard unit normal of this "patch" is double-valued at $(1,0,0)$.

(iii) $U = \{(t,\theta) \mid -\frac{1}{2} < t < \frac{1}{2},\ 0 \leq \theta < 2\pi\}$

Not open in $\mathbb{R}^2$.

(iv) Use only patch $\sigma_1 : U_1 \to \mathbb{R}^3$.

Doesn't cover $(1,0,0)$.

Lecture 9: The first fundamental form 1FF

The first fundamental form (1FF) of a surface

Definition (First fundamental form)

The 1FF of a surface S at p is the restriction of the inner product in $\mathbb{R}^3$ (i.e., the dot product) to vectors in T_pS:

$$\langle \mathbf{u}, \mathbf{v} \rangle_{p,S} = \mathbf{u} \cdot \mathbf{v} \ , \ \mathbf{u}, \mathbf{v} \in T_pS \subset \mathbb{R}^3 .$$

- Usually just write $\langle \mathbf{u}, \mathbf{v} \rangle$ (omit subscripts p, S when no confusion can occur).
- Older books sometimes use a roman I, as in $I(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle$. We will use $\mathcal{F}_I$.
- In Riemannian geometry, the 1FF is called the *induced metric* on S.
- Can consider the 1FF to be the map that associates to each $p \in S$ an inner product $\langle \cdot, \cdot \rangle_p$ on T_pS at $p \in S$.

The 1FF on a single surface patch

- The 1FF $\langle \cdot, \cdot \rangle_{p,S}$ is a *symmetric bilinear form*.
- Surface patch $X : U \to \mathbb{R}^3$ containing p.
- Basis $\{X_u, X_v\}$ for T_pS, so $\mathbf{v} \in T_pS \implies \mathbf{v} = \alpha X_u + \beta X_v$.

$$\langle \mathbf{v}, \mathbf{v} \rangle_X = \alpha^2 \langle X_u, X_u \rangle + 2\alpha\beta \langle X_u, X_v \rangle + \beta^2 \langle X_v, X_v \rangle.$$

- Notation: When expressed in the above basis, we write $\langle \cdot, \cdot \rangle_{p,X}$.
- Write

$$\begin{aligned} E &= \langle X_u, X_u \rangle = \|X_u\|^2 , \\ F &= \langle X_u, X_v \rangle = X_u \cdot X_v , \implies \langle \mathbf{v}, \mathbf{v} \rangle = E\alpha^2 + 2F\alpha\beta + G\beta^2 \\ G &= \langle X_v, X_v \rangle = \|X_v\|^2 , \end{aligned}$$

- Define the linear maps *du* and *dv* (scalar projection) by
  - $du(\mathbf{v}) = \alpha$
  - $dv(\mathbf{v}) = \beta$
  - Then $\langle \mathbf{v}, \mathbf{v} \rangle_X = E\, du(\mathbf{v})\, du(\mathbf{v}) + 2F\, du(\mathbf{v})\, dv(\mathbf{v}) + G\, dv(\mathbf{v})\, dv(\mathbf{v})$.

Explicit form for the 1FF on a patch

- Patch $X : U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$.
- From last slide, can write the 1FF as

$$\langle \mathbf{v}, \mathbf{v} \rangle_X = E\, du(\mathbf{v})\, du(\mathbf{v}) + 2F\, du(\mathbf{v})\, dv(\mathbf{v}) + G\, dv(\mathbf{v})\, dv(\mathbf{v}).$$

- Often write $\langle \mathbf{v}, \mathbf{v} \rangle = (E\, du\, du + 2F\, du\, dv + G\, dv\, dv)\,(\mathbf{v}, \mathbf{v})$ or simply

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

  This notation is sometimes called the *line element form* of the 1FF.

- Matrix for $\langle \cdot, \cdot \rangle_{p,X}$ in $\{X_u, X_v\}$ basis:

$$[\mathcal{F}_I] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Arclength and line element form

- The 1FF can be used to find the arclength of a space curve γ lying on surface S.
- Say $\gamma : [a,b] \to \mathbb{R}^3$ lies within one patch $X : U \to \mathbb{R}^3$ of S, so $\gamma(t) = X(u(t), v(t))$.
- Then $\dot{\gamma} = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt} = \dot{u}X_u + \dot{v}X_v$.
- $\langle \dot{\gamma}, \dot{\gamma} \rangle = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$ where E, F, G are evaluated at $\gamma(t)$.
- Arclength

$$s = \int ds = \int_a^b \frac{ds}{dt}dt = \int_a^b \|\dot{\gamma}(t)\|\, dt = \int_a^b \langle \dot{\gamma}, \dot{\gamma} \rangle^{1/2} dt$$

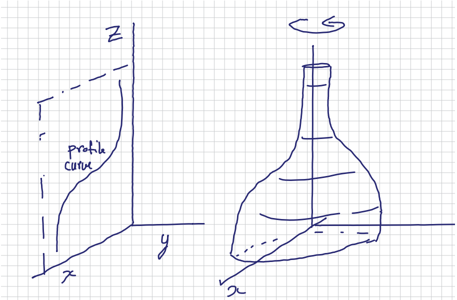
$$= \int_a^b \sqrt{E(\gamma(t))\dot{u}^2 + 2F(\gamma(t))\dot{u}\dot{v} + G(\gamma(t))\dot{v}^2}\, dt.$$

Why are the projections called du, dv?

- Two overlapping patches $X : U \to \mathbb{R}^3$ and $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ for S.
- $\mathbf{v} = \alpha X_u + \beta X_v = \tilde{\alpha}\tilde{X}_{\tilde{u}} + \tilde{\beta}\tilde{X}_{\tilde{v}}$ with $X(u,v) = \tilde{X}(\tilde{u},\tilde{v})$.
- $\tilde{X}_{\tilde{u}} = \frac{\partial \tilde{X}}{\partial \tilde{u}} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial \tilde{u}} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \tilde{u}} = X_u \frac{\partial u}{\partial \tilde{u}} + X_v \frac{\partial v}{\partial \tilde{u}}$.
- $\tilde{X}_{\tilde{v}} = \frac{\partial \tilde{X}}{\partial \tilde{v}} = \frac{\partial X}{\partial u}\frac{\partial u}{\partial \tilde{v}} + \frac{\partial X}{\partial v}\frac{\partial v}{\partial \tilde{v}} = X_u \frac{\partial u}{\partial \tilde{v}} + X_v \frac{\partial v}{\partial \tilde{v}}$.
- So $\mathbf{v} = \alpha X_u + \beta X_v = \tilde{\alpha}\tilde{X}_{\tilde{u}} + \tilde{\beta}\tilde{X}_{\tilde{v}} = \left(\tilde{\alpha}\frac{\partial u}{\partial \tilde{u}} + \tilde{\beta}\frac{\partial u}{\partial \tilde{v}}\right) X_u + \left(\tilde{\alpha}\frac{\partial v}{\partial \tilde{u}} + \tilde{\beta}\frac{\partial v}{\partial \tilde{v}}\right) X_v$.
- X_u-component: $\alpha = du(\mathbf{v}) = \tilde{\alpha}\frac{\partial u}{\partial \tilde{u}} + \tilde{\beta}\frac{\partial u}{\partial \tilde{v}} = \frac{\partial u}{\partial \tilde{u}} d\tilde{u}(\mathbf{v}) + \frac{\partial u}{\partial \tilde{v}} d\tilde{v}(\mathbf{v})$.
- X_v-component: $\beta = dv(\mathbf{v}) = \tilde{\alpha}\frac{\partial v}{\partial \tilde{u}} + \tilde{\beta}\frac{\partial v}{\partial \tilde{v}} = \frac{\partial v}{\partial \tilde{u}} d\tilde{u}(\mathbf{v}) + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}(\mathbf{v})$.
- This gives an easy mnemonic for the transformation rules $(u,v) \mapsto (\tilde{u},\tilde{v}) = \Phi^{-1}(u,v)$; compare to chain rule for differentials, which gives:

$$du = \frac{\partial u}{\partial \tilde{u}} d\tilde{u} + \frac{\partial u}{\partial \tilde{v}} d\tilde{v} \text{ and } dv = \frac{\partial v}{\partial \tilde{u}} d\tilde{u} + \frac{\partial v}{\partial \tilde{v}} d\tilde{v}.$$

Example: Surfaces of revolution



- $\gamma(u) = (f(u), 0, g(u))$, $f(u) \geq 0$, is called the *profile curve*.
- Unit speed if $\dot{f}^2(u) + \dot{g}^2(u) = 1$.
- Surface $X(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$.

1FF of a surface of revolution

- Surface $X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$.
- Basis for T_pS:

$$X_u = \left(\dot{f}(u)\cos v, \dot{f}(u)\sin v, \dot{g}(u)\right)\ ,\ X_v = (-f(u)\sin v, f(u)\cos v, 0).$$

- $E = \|X_u\|^2 = \dot{f}^2(u) + \dot{g}^2(u) = 1$ if profile curve is unit speed.
- $F = X_u \cdot X_v = -f\dot{f}\cos v \sin v + f\dot{f}\sin v\cos v = 0$.
- $G = \|X_v\|^2 = f^2(u)$.
- 1FF is $ds^2 = du^2 + f^2(u)dv^2$ or in matrix form $[\mathcal{F}_I] = \begin{bmatrix} 1 & 0 \\ 0 & f^2(u) \end{bmatrix}$.

Example: a sphere

- Profile curve: semi-circle $x = \sqrt{a^2 - z^2}$, $a > 0$.
- Unit speed parametrization: $\gamma(u) = \left(a \sin \frac{u}{a}, 0, a \cos \frac{u}{a}\right)$, $u \in [0, a\pi]$.
- Surface of revolution $X(u, v) = a \left(\sin \frac{u}{a} \cos v, \sin \frac{u}{a} \sin v, \cos \frac{u}{a}\right)$, $u \in [0, a\pi]$, $v \in [0, 2\pi)$, is a sphere of radius a.
- Then $f(u) = a \sin \frac{u}{a}$, $g(u) = a \cos \frac{u}{a}$.
- 1FF is $ds^2 = du^2 + f^2(u)dv^2 = du^2 + a^2 \sin^2 \frac{u}{a} dv^2$.
- Looks more familiar if we let $\phi = \frac{u}{a} \in [0, \pi]$, $\theta = v \in [0, 2\pi)$:

$$ds^2 = a^2 \left(d\phi^2 + \sin^2 \phi d\theta^2\right).$$

Pullbacks

- Say $p \in S_1$ and $\mathbf{v}, \mathbf{w} \in T_pS$.
- Let $\tilde{\mathbf{v}} = D_pf(\mathbf{v})$, $\tilde{\mathbf{w}} = D_pf(\mathbf{w})$ be push-forwards.
- Let $\langle \cdot, \cdot \rangle_{q,S_2}$ be the 1FF on S_2, $q = f(p)$.

Definition (pullback metric)

We define an inner product $f^*\langle \cdot, \cdot \rangle_{p,S_1}$, called the *pullback* of $\langle \cdot, \cdot \rangle_{q,S_2}$ by f, by

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_{p,S_1} = \langle \tilde{\mathbf{v}}, \tilde{\mathbf{w}} \rangle_{f(p),S_2} = \langle D_pf\mathbf{v}, D_pf\mathbf{w} \rangle_{f(p),S_2}.$$

Notation: When comparing 1FFs on two surfaces, say S_1 and S_2, we will sometimes use parentheses rather than angle brackets to distinguish them; e.g., $\langle \cdot, \cdot \rangle_{S_1}$ and $(\cdot, \cdot)_{S_2}$.

Local isometry and pullbacks

Definition (Local isometry)

Let $f : S_1 \to S_2$ be a smooth map between surfaces. If for every curve $\gamma : I \to \mathbb{R}^3$ in S_1, its image $\tilde{\gamma} = f \circ \gamma : I \to \mathbb{R}^3$ in S_2 has the same arclength, then f is a *local isometry*, and we say that S_1 and S_2 are *locally isometric*.

- Let $\langle \cdot, \cdot \rangle_{p,S_1}$ be the 1FF on S_1.
- Let $(\cdot, \cdot)_{q,S_2}$ be the 1FF on S_2, $q = f(p)$, for some smooth map $f : S_1 \to S_2$.

Theorem

Say that $f^*(\cdot, \cdot)_{p,S_1} = \langle \cdot, \cdot \rangle_{p,S_1}$ *for all* $p \in S_1$. *If* $\gamma : I \to \mathbb{R}^3$ *is a curve in* S_1 *with arclength* s *and* $\tilde{\gamma} = f \circ \gamma : I \to \mathbb{R}^3$ *is its image curve in* S_2 *with arclength* $\tilde{s}$, *then* $s = \tilde{s}$ *and so* f *is a local isometry.*

Proof

- Arclength of $\gamma : [t_0, t_1] \to \mathbb{R}^3$ in S_1:

$$s = \int_{t_0}^{t_1} \langle \dot{\gamma}, \dot{\gamma} \rangle_{\gamma(t), S_1}^{1/2} \, dt.$$

- Arclength of $\tilde{\gamma} = f \circ \gamma : [t_0, t_1] \to \mathbb{R}^3$ in S_2:

$$\tilde{s} = \int_{t_0}^{t_1} \left(Df(\dot{\gamma}), Df(\dot{\gamma}) \right)_{\tilde{\gamma}(t), S_2}^{1/2} \, dt = \int_{t_0}^{t_1} \sqrt{f^* \left(\dot{\gamma}, \dot{\gamma} \right)_{\gamma(t), S_1}} dt.$$

- But if $f^* \left(\cdot, \cdot \right)_{p, S_1} = \langle \cdot, \cdot \rangle_{p, S_1}$ for all $p \in S_1$, these two expressions are clearly equal.

The converse is also true, but harder to prove so we'll skip the proof:

Theorem

If $s = \tilde{s}$ for all curves γ is S_1 and their images $\tilde{\gamma} = f \circ \gamma$ in S_2, then $\langle \dot{\gamma}, \dot{\gamma} \rangle_p = f^* \left(\dot{\gamma}, \dot{\gamma} \right)_p$ for all $p \in S_1$.

Moreover...

Theorem

$\langle \mathbf{v}, \mathbf{v} \rangle = f^*(\mathbf{v}, \mathbf{v})$ for all $\mathbf{v} \in T_pS$ iff $\langle \mathbf{v}, \mathbf{w} \rangle = f^*(\mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in T_pS$.

Proof.

If $\langle \mathbf{v}, \mathbf{v} \rangle = f^*(\mathbf{v}, \mathbf{v})$ for all $\mathbf{v} \in T_pS$, then compute

$$\begin{aligned} & \langle \mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w} \rangle = f^*(\mathbf{v}+\mathbf{w}, \mathbf{v}+\mathbf{w}) \\ \implies & \langle \mathbf{v}, \mathbf{v} \rangle + 2\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle = f^*(\mathbf{v}, \mathbf{v}) + 2f^*(\mathbf{v}, \mathbf{w}) + f^*(\mathbf{w}, \mathbf{w}) \\ \implies & 2\langle \mathbf{v}, \mathbf{w} \rangle = 2f^*(\mathbf{v}, \mathbf{w}) \end{aligned}$$

and we are done. This is an example of the *polarization identity*. □

Theorem

A smooth map $f : S_1 \to S_2$ is a local isometry if and only if the symmetric bilinear forms $\langle \cdot, \cdot \rangle_p$ and $f^*(\cdot, \cdot)_p$ on T_pS_1 are equal for all $p \in S_1$.

Local isometries and the 1FF

If our smooth map f is a diffeomorphism (i.e., if it has a smooth inverse), then

Corollary

A local diffeomorphism $f : S_1 \to S_2$ is a local isometry if and only if, for any surface patch X for S_1, the patches X and $\tilde{X} = f \circ X$ of S_2 have the same 1FF:

$$\langle \cdot, \cdot \rangle_X = f^*(\cdot, \cdot)_{\tilde{X},p}.$$

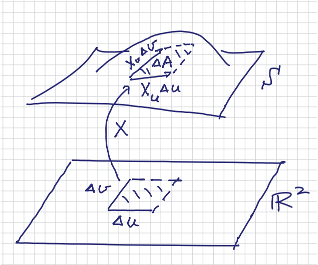
In other words, if f is a local diffeomorphism from S_1 to S_2, the geometry encoded in the 1FF is the same about $p \in S_1$ as it is $f(p) \in S_2$.
Proof: See text p 128.

Lecture 10: Equiareal maps

Area of a surface

- Surface S, parameters $(u, v) \in U$.
- Surface patch $X : U \to \mathbb{R}^3$.
- Basis $\{X_u, X_v\}$ for T_pS.
- Small parallelogram of area $\Delta u \Delta v$ in U.
- Image has sides $X_u\Delta u$, $X_v\Delta v$ in S and area $\Delta A = \|X_u \times X_v\| \Delta u \Delta v$.
- Let R be a region in U and $\mathcal{R} = X(R)$ be its image in S. The area of $\mathcal{R}$ is

$$A_X(\mathcal{R}) = \int_R dA = \int_U \|X_u \times X_v\| \, dudv.$$



Area "well-defined"

Theorem

$A_X(\mathcal{R})$ does not depend on the choice of regular coordinate patch $X : U \to \mathbb{R}^3$

In consequence, we can simply write $A(\mathcal{R})$, without a subscript

Proof.

If $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ is another regular coordinate patch covering $\mathcal{R}$ and $\phi : \tilde{U} \to U$ is smooth, we already know that

$$\begin{aligned}
&\tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} = (\det J(\Phi)) X_u \times X_v \\
\implies & \left\| \tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} \right\| = |(\det J(\Phi))| \, \| X_u \times X_v \| \\
\implies & A_{\tilde{X}} = \int_{\tilde{U}} \left\| \tilde{X}_{\tilde{u}} \times \tilde{X}_{\tilde{v}} \right\| d\tilde{u} d\tilde{v} = \int_{\tilde{U}} \| X_u \times X_v \| \, |(\det J(\Phi))| \, d\tilde{u} d\tilde{v} \\
& \qquad = \int_U \| X_u \times X_v \| \, du dv \text{ by the change of variables formula} \\
& \qquad = A_X .
\end{aligned}$$

Local form of area element

Theorem

In a patch X the area element $dA = dA_X = \|X_u \times X_v\|\, dudv$ can be written as $dA = \sqrt{\det(\mathcal{F}_I)} dudv$, where $\mathcal{F}_I$ is the matrix for the 1FF.

Proof.

$$\begin{aligned} \|X_u \times X_v\|^2 &= (X_u \times X_v) \cdot (X_u \times X_v) \\ &= (X_u \cdot X_u)(X_v \cdot X_v) - (X_u \cdot X_v)^2 \text{ by a standard identity} \\ &= EG - F^2 = \det(\mathcal{F}_I). \end{aligned}$$

☐

Area of surface region R covered by a single patch $X : U \to \mathbb{R}^3$:

$$A(R) = \int_R dA = \int_U \sqrt{\det(\mathcal{F}_I)} dudv.$$

Equiareal maps

Definition

A local diffeomorphism $f : S_1 \to S_2$ is *equiareal* if it takes each region $\mathcal{R}_1 \subset S_1$ to a region $\mathcal{R}_2 = f(\mathcal{R}_1) \subseteq S_2$ of the same area.

Theorem

$f : S_1 \to S_2$ is equiareal iff for any surface patch $X : U \to \mathbb{R}^3$ on S_1, the 1FFs

- *$E_1 du^2 + 2F_1 dudv + G_1 dv^2$ if the patch X on S_1 and*
- *$E_2 du^2 + 2F_2 dudv + G_2 dv^2$ if the patch $f \circ X$ on S_2*

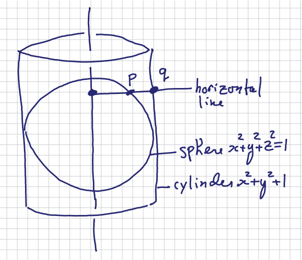
satisfy $E_1 G_1 - F_1^2 = E_2 G_2 - F_2^2$.

Proof.

$E_1 G_1 - F_1^2 = \det\left(\mathcal{F}_{I_{S_1}}\right)$ and $E_2 G_2 - F_2^2 = \det\left(\mathcal{F}_{I_{S_2}}\right)$ and by the previous theorem, the area elements equal iff these determinants are equal. □

Archimedes's equiareal map

- Unit sphere $x^2 + y^2 + z^2 = 1$ (denoted S_1^2) and unit vertical cylinder $x^2 + y^2 = 1$.
- Let p and q lie on horizontal radial line, with p on the sphere and q on the cylinder.
- This defines a map f taking $p \in S_1^2$, except the poles, to some q on the cylinder.
- If $p = (x, y, z)$ then
  $$q = f(p) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z \right).$$
- Archimedes's theorem: f is an equiareal diffeomorphism.



Proof of Archimedes's theorem

- Need an atlas for S_1^2 minus the poles.
- Take $X(\theta,\varphi) := (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, defined on two open sets: $U_1 = \left\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \varphi < 2\pi\right\}$ and $U_2 = \left\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}, -\pi < \varphi < \pi\right\}$.
- Two patches, same *formula* for $X : U_1 \to \mathbb{R}^3$ and $X : U_2 \to \mathbb{R}^3$.
- Basis for tangent space: $X_\theta = (-\sin\theta\cos\varphi, -\sin\theta\sin\varphi, \cos\theta)$, $X_\varphi = (-\cos\theta\sin\varphi, \cos\theta\cos\varphi, 0)$.
- Then $E_1 = \|X_\theta\|^2 = 1$, $F_1 = X_\theta \cdot X_\varphi = 0$, $G_1 = \|X_\varphi\|^2 = \cos^2\theta$.
- Then the determinant of the 1FF is $\begin{vmatrix} 1 & 0 \\ 0 & \cos^2\theta \end{vmatrix} = \cos^2\theta$.

Proof of Archimedes's theorem continued

- Since $f(x,y,z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z\right)$ and $X(\theta,\phi) = (x,y,z) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$, then

$$(f\circ X)(\theta,\phi) = \left(\frac{\cos\theta\cos\varphi}{\cos\theta}, \frac{\cos\theta\sin\varphi}{\cos\theta}, \sin\theta\right) = (\cos\varphi, \sin\varphi, \sin\theta)$$

- Then $(f\circ X)_\theta = (0,0,\cos\theta)$ and $(f\circ X)_\varphi = (-\sin\varphi, \cos\varphi, 0)$.
- Then $E_2 = \|(f\circ X)_\theta\|^2 = \cos^2\theta$, $F_2 = (f\circ X)_\theta \cdot (f\circ X)_\varphi = 0$, $G_2 = \|(f\circ X)_\varphi\|^2 = 1$.
- Then the determinant of the 1FF is $\begin{vmatrix} \cos^2\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos^2\theta$.
- This determinant equals the one on the last slide. This completes the proof.

Corollary: spherical triangles

- Consider a 2-dimensional unit sphere $\mathbb{S}^2$ defined by $x^2 + y^2 + z^2 = 1$.
- A *great circle* is the curve of intersection of this sphere with a plane that contains $(0, 0, 0)$.
- A spherical triangle is a triangle on $\mathbb{S}^2$ whose sides are segments of great circles that meet at 3 vertices.

Theorem

If a spherical triangle on the unit sphere $\mathbb{S}$ has interior angles α, β, and γ, then the area of the spherical triangle is $\alpha + \beta + \gamma - \pi$.

Proof.

See text pp 145–147. Uses Archimedes's theorem. ☐

Lecture 11: The second fundamental form 2FF

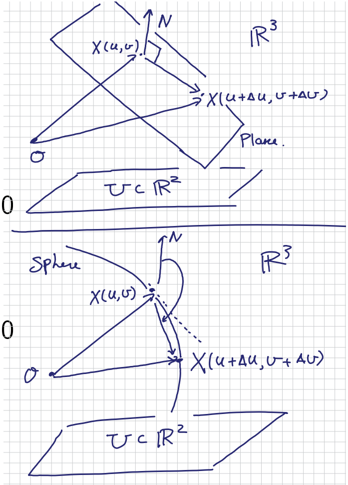
Curvature of surfaces

- Plane:

$$\mathbf{N} \cdot (X(u + \Delta u, v + \Delta v) - X(u, v)) = 0$$

- Sphere:

$$\mathbf{N} \cdot (X(u + \Delta u, v + \Delta v) - X(u, v)) \neq 0$$



- Taylor's theorem:

$$X(u+\Delta u, v+\Delta v) = X(u,v) + X_u(u,v)\Delta u + X_v(u,v)\Delta v \\ + \frac{1}{2}\left[X_{uu}(\Delta u)^2 + 2X_{uv}\Delta u\Delta v + X_{vv}(\Delta v)^2\right] + \dots$$

  so

$$\mathbf{N}\cdot\left(X(u+\Delta u, v+\Delta v) - X(u,v)\right) = \frac{1}{2}\left[\mathbf{N}\cdot X_{uu}(\Delta u)^2 + 2\mathbf{N}\cdot X_{uv}\Delta u\Delta v \\ + \mathbf{N}\cdot X_{vv}(\Delta v)^2\right] + \dots$$

- We define: $L := \mathbf{N}\cdot X_{uu}$, $M = \mathbf{N}\cdot X_{uv}$, $N := \mathbf{N}\cdot X_{vv}$. The above equation is

$$\mathbf{N}\cdot\left(X(u+\Delta u, v+\Delta v) - X(u,v)\right) = \frac{1}{2}\left[L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\right] + \dots$$

- Compare to unit speed curve $\gamma(t)$ in $\mathbb{R}^2$:

$$\gamma(t+\Delta t) = \gamma(t) + \dot{\gamma}(t)\Delta t + \frac{1}{2}\ddot{\gamma}(t)(\Delta t)^2 + \dots$$

$$\implies \mathbf{N}\cdot\left(\gamma(t+\Delta t) - \gamma(t)\right) = \frac{1}{2}\kappa_S(\Delta t)^2 + \dots$$

- So $L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2$ is a "surface version" of $\kappa_S(\Delta t)^2$.

The Second Fundamental Form (2FF)

Definition (Second Fundamental Form of a surface patch)

The 2FF of a surface patch $X : U \to \mathbb{R}^3$ is the map $\langle\langle\cdot,\cdot\rangle\rangle_X : T_pS \times T_pS \to \mathbb{R}$ defined in line element form to be

$$Ldu^2 + Mdudv + Mdvdu + Ndv^2,$$

so that

$$\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_X = Ldu(\mathbf{v})du(\mathbf{w}) + Mdu(\mathbf{v})dv(\mathbf{w}) + Mdv(\mathbf{v})du(\mathbf{w}) + Ndv(\mathbf{v})dv(\mathbf{w}).$$

We also write the (symmetric) matrix form as $[\mathcal{F}_{II}]$ where

$$\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_X = [\mathbf{v}]^T [\mathcal{F}_{II}] [\mathbf{w}] = [\mathbf{v}]^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} [\mathbf{w}].$$

The 2FF is also called the *extrinsic curvature*.

Transformation law

- Patches $X : U \to \mathbb{R}^3$ and $\tilde{X} : \tilde{U} \to \mathbb{R}^3$ with $X(u,v) = \tilde{X}(\tilde{u},\tilde{v})$.
- Chain rule $\tilde{X}_{\tilde{u}} = X_u \frac{\partial u}{\partial \tilde{u}} + X_v \frac{\partial v}{\partial \tilde{u}}$.
- Then $\tilde{X}_{\tilde{u}\tilde{u}} = X_{uu} \left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + X_{uv} \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{u}} + X_u \frac{\partial^2 u}{\partial \tilde{u}^2} + X_{vu} \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + X_{vv} \left(\frac{\partial v}{\partial \tilde{u}}\right)^2 + X_v \frac{\partial^2 v}{\partial \tilde{u}^2}$.
- Then (using $\tilde{\mathbf{N}} = \pm \mathbf{N} = \frac{\det(J)}{|\det J|} \mathbf{N}$) we get

$$\begin{aligned}\tilde{L} = \tilde{\mathbf{N}} \cdot \tilde{X}_{\tilde{u}\tilde{u}} &= \pm \mathbf{N} \cdot \left[X_{uu} \left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + X_{uv} \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{u}} + X_u \frac{\partial^2 u}{\partial \tilde{u}^2} + X_{vu} \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + X_{vv} \left(\frac{\partial v}{\partial \tilde{u}}\right)^2 + X_v \frac{\partial^2 v}{\partial \tilde{u}^2} \right] \\ &= \pm \left[L \left(\frac{\partial u}{\partial \tilde{u}}\right)^2 + M \frac{\partial v}{\partial \tilde{u}} \frac{\partial u}{\partial \tilde{u}} + 0 + M \frac{\partial u}{\partial \tilde{u}} \frac{\partial v}{\partial \tilde{u}} + N \left(\frac{\partial v}{\partial \tilde{u}}\right)^2 + 0 \right]\end{aligned}$$

  In the last line, we used that $\mathbf{N} \perp T_pS = \text{Span}\{X_u, X_v\}$.
- This is one component of the matrix equation

$$\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \pm \begin{bmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial u}{\partial \tilde{v}} \\ \frac{\partial v}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{v}} \end{bmatrix} = \pm [J]^T [\mathcal{F}_{II}][J].$$

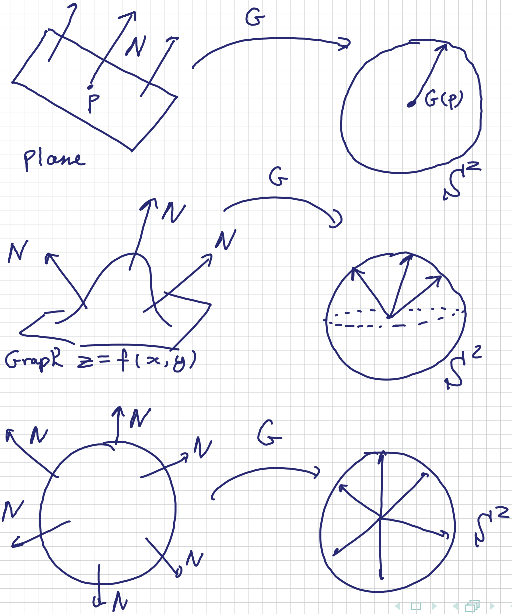
- Transformation law for 2FF: $\left[\widetilde{\mathcal{F}}_{II}\right] = \frac{\det(J)}{|\det J|} [J]^T [\mathcal{F}_{II}] [J]$.

Gauss and Weingarten maps

Definition

The Gauss map G of an oriented surface S maps each $p \in S$ to the unit normal $\mathbf{N}$ at p.

- The set of all unit vectors based at the origin in $\mathbb{R}^3$ is a unit sphere $\mathbb{S}^2 = S_1^2$. Therefore we may regard the Gauss map as a smooth map $G : S \to \mathbb{S}^2$.
- The image of the Gauss map of a plane is a single point in $\mathbb{S}^2$.
- The image of the Gauss map of a graph is contained in the upper hemisphere.
- The image of the Gauss map of a sphere is contains every point of $\mathbb{S}^2$.
- Exercise: What does the gauss map of a torus look like? (Answer: it covers $\mathbb{S}^2$ twice.)


N
P
G
G(p)
Plane
S^2
N
N
N
N
G
Graph z = f(x,y)
S^2
N
N
N
N
N
N
N
G
S^2

The Derivative of G

- Diffeomorphism $G : S \to \mathbb{S}^2$.
- Derivative at $p \in S$ is $D_pG : T_pS \to T_{G(p)}\mathbb{S}^2$.
- Measures the change in $\mathbf{N}$ as $p = X(u, v) \in S$ changes.
- $\Delta\mathbf{N} = (D_pG)(\Delta X)$.
- Since $\|\mathbf{N}\| = 1$, then $D_pG(\Delta X) \perp G(p) = \mathbf{N}_p$. So $D_pG(\Delta X) \in T_pS$.
- Therefore, $D_pG : T_pS \to T_pS$.

The Weingarten map

Definition (Weingarten map)

We define the Weingarten map $W_{p,S} : T_pS \to T_pS$ of the surface S at $p \in S$ to be the linear map $W_{p,S} = -D_pG$.

We note that $W_{p,S}$ is an *operator* or *endomorphism* since it maps T_pS to itself. Therefore $W_{p,S}$ can have eigenvalues/eigenvectors.

Definition (2FF of a surface)

The *second fundamental form of a surface* S at $p \in S$ is the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{p,S} : T_pS \times T_pS \to \mathbb{R}$ such that

$$\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,S} = \langle W_{p,S}(\mathbf{v}), \mathbf{w} \rangle, \ \mathbf{v}, \mathbf{w} \in T_pS.$$

How to compute the 2FF of a surface

Theorem

1. *$\langle\langle \cdot, \cdot \rangle\rangle_{p,S}$ is bilinear.*
2. *$\langle\langle \cdot, \cdot \rangle\rangle_{p,S}$ is symmetric: $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,S} = \langle\langle \mathbf{w}, \mathbf{v} \rangle\rangle_{p,S}$.*
3. *On a surface patch $X : U \to \mathbb{R}^3$, $\langle\langle \cdot, \cdot \rangle\rangle_{p,S} = \langle\langle \cdot, \cdot \rangle\rangle_{p,X}$.*

- The first line above means that for $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in T_pS$, then
  - $\langle\langle a\mathbf{u} + b\mathbf{v}, \mathbf{w} \rangle\rangle_{p,S} = a\langle\langle \mathbf{u}, \mathbf{w} \rangle\rangle_{p,S} + b\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,S}$ and
  - $\langle\langle \mathbf{w}, a\mathbf{u} + b\mathbf{v} \rangle\rangle_{p,S} = a\langle\langle \mathbf{w}, \mathbf{u} \rangle\rangle_{p,S} + b\langle\langle \mathbf{w}, \mathbf{v} \rangle\rangle_{p,S}$.

  This follows form the linearity of D_pG.
- If the third line is true, then the second line follows from the symmetry of $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$.
- So we must prove the third statement.

But first, interpret statement 2 of theorem

- By the definition of the 2FF of a surface we have $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle_{p,S} = \langle W(\mathbf{u}), \mathbf{v} \rangle_{p,S}$.
- We can also write $\langle\langle \mathbf{v}, \mathbf{u} \rangle\rangle_{p,S} = \langle W(\mathbf{v}), \mathbf{u} \rangle_{p,S}$.
- Since the 1FF is symmetric, the last line gives $\langle\langle \mathbf{v}, \mathbf{u} \rangle\rangle_{p,S} = \langle \mathbf{u}, W(\mathbf{v}) \rangle_{p,S}$.
- Therefore, the left-hand sides of the first and third lines equal if and only if their right-hand sides equal:

$$\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle_{p,S} = \langle\langle \mathbf{v}, \mathbf{u} \rangle\rangle_{p,S} \Leftrightarrow \langle W(\mathbf{u}), \mathbf{v} \rangle_{p,S} = \langle \mathbf{u}, W(\mathbf{v}) \rangle_{p,S}$$

- The left-hand equation expresses the *symmetry* of the 2FF. The right-hand side expresses the *self-adjointness of W with respect to the inner product* that is the 1FF.
- The eigenvalues of a self-adjoint operator are always real numbers.

Proving part 3

Step 1: Rewrite the surface patch 2FF $Ldu^2 + 2Mdudv + Ndv^2$.

- Patch $X : U \to \mathbb{R}^3$ with standard normal $\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|}$.
- Then $\mathbf{N} \cdot X_u = 0$ and $\mathbf{N} \cdot X_v = 0$.
- Differentiate: $\mathbf{N}_u \cdot X_u + \mathbf{N} \cdot X_{uu} = 0 = \mathbf{N}_v \cdot X_u + \mathbf{N} \cdot X_{uv}$.
- And $\mathbf{N}_u \cdot X_v + \mathbf{N} \cdot X_{vu} = 0 = \mathbf{N}_v \cdot X_v + \mathbf{N} \cdot X_{vv}$.
- Using $L = \mathbf{N} \cdot X_{uu}$, $M = \mathbf{N} \cdot X_{uv}$, and $N = \mathbf{N} \cdot X_{vv}$, we now get

$$\begin{aligned} L &= -\mathbf{N}_u \cdot X_u, \\ M &= -\mathbf{N}_u \cdot X_v = -\mathbf{N}_v \cdot X_u, \\ N &= -\mathbf{N}_v \cdot X_v. \end{aligned}$$

- Because $\mathbf{N}_u, \mathbf{N}_v, X_u, X_v \in T_pS$, can replace dot product by $\langle \cdot, \cdot \rangle$ in these expressions.

Step 2: Rewrite the surface 2FF $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,S} = \langle W_{p,S}(\mathbf{v}), \mathbf{w} \rangle$, $W_{p,S} = -D_pG$, $\mathbf{v}, \mathbf{w} \in T_pS$.

- Choose a patch X containing $p = X(u_0, v_0)$. Then:

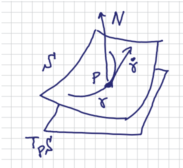
$$\begin{aligned} W_{p,S}(X_u) &= -D_pG(X_u) = -\frac{d}{du}\Big|_{u=u_0} G(X(u, v_0)) \\ &= -\frac{d}{du}\Big|_{u=u_0} \mathbf{N}(u, v_0) = -\mathbf{N}_u(u_0, v_0) \\ \implies \langle W_{p,S}(X_u), X_u \rangle &= -\langle \mathbf{N}_u, X_u \rangle = L \\ \implies L = \langle W_{p,S}(X_u), X_u \rangle. \end{aligned}$$

- Similar calculations give $M = \langle W_{p,S}(X_v), X_u \rangle = \langle W_{p,S}(X_u), X_v \rangle$ and $N = \langle W_{p,S}(X_v), X_v \rangle$.
- We conclude that when $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,S}$ is restricted to a patch X and its components are computed, they equal the components of the surface patch 2FF $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle_{p,X}$.

Lecture 12: Normal and geodesic curvatures

Curves on surfaces

- γ a unit speed curve.
- $\dot{\gamma} \cdot \dot{\gamma} = 1.$
- $\dot{\gamma} \cdot \ddot{\gamma} = 0.$
- Then $\ddot{\gamma} \in \text{Span}\,\{\mathbf{N}, \mathbf{N} \times \dot{\gamma}\}$
- Note: $\mathbf{N} \times \dot{\gamma}$ is a unit vector.
- $\ddot{\gamma} = \kappa_N \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}$ where κ_N and κ_g are coefficients in this linear combination.



The components κ_N and κ_g of $\ddot{\gamma}$

- γ is a unit speed curve in surface S.
- $\ddot{\gamma} = \kappa_N \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}$.
- Then $\kappa_N = \ddot{\gamma} \cdot \mathbf{N}$ is called the *normal curvature* of γ. It is due to the bending of the surface S.
- $\kappa_g = \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})$ is called the *geodesic curvature* of γ, due to bending (i.e., acceleration) of curve within S.
- Since $\mathbf{N}$ and $\mathbf{N} \times \dot{\gamma}$ are unit vectors and are perpendicular to each other, then

$$\|\ddot{\gamma}\|^2 = (\kappa_N \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}) \cdot (\kappa_N \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}) = \kappa_N^2 + \kappa_g^2.$$

- But γ is a space curve, so if has curvature κ given by $\|\ddot{\gamma}\| = \kappa$. Then we have the relation between curvature, geodesic curvature, and normal curvature:

$$\kappa^2 = \kappa_N^2 + \kappa_g^2.$$

The Frenet frame again

- Recall *principal normal* **n** to unit speed curve γ.

$$\mathbf{n} = \frac{1}{\kappa}\ddot{\gamma}.$$

- Principal normal might not lie in T_pS, where γ lies in S.
- Let **N** be normal to surface S.
- Define ψ to be angle between **n** and **N**, so $\mathbf{n} \cdot \mathbf{N} = \cos\psi$.
- Then

$$\begin{aligned} \kappa\mathbf{n} = \ddot{\gamma} &= \kappa_N\mathbf{N} + \kappa_g\mathbf{N} \times \dot{\gamma} \\ \implies \kappa\mathbf{n} \cdot \mathbf{N} &= \kappa_N\mathbf{N} \cdot \mathbf{N} + \kappa_g\left(\mathbf{N} \times \dot{\gamma}\right) \cdot \mathbf{N} \\ \implies \kappa\cos\psi &= \kappa_N. \end{aligned}$$

- Then $\kappa_N = \kappa\cos\psi$ and $\kappa_g = \kappa\sin\psi$ (since $\kappa^2 = \kappa_N^2 + \kappa_g^2$).

κ_N is a property of the surface, not the curve

- Normal to S at p is $\mathbf{N} = G(p)$ (Gauss map).
- Curve $\gamma(t)$ in S passes through $p = \gamma(0)$.
- $\dot{\mathbf{N}} = \frac{d}{dt}\Big|_{t=0} G = (D_p G)(\dot\gamma) = -W(\dot\gamma)$.
- Now we can compute $\kappa_N = \mathbf{N} \cdot \ddot\gamma = \frac{d}{dt}(\mathbf{N} \cdot \dot\gamma) - \dot{\mathbf{N}} \cdot \dot\gamma = -\dot{\mathbf{N}} \cdot \dot\gamma = W(\dot\gamma) \cdot \dot\gamma = \langle W(\dot\gamma), \dot\gamma \rangle$.
- Finally, recall the definition of the 2FF: $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle = \langle W(\mathbf{v}), \mathbf{w} \rangle$.
- Then $\kappa_N = \langle\langle \dot\gamma, \dot\gamma \rangle\rangle$.
- Surface patch form: If $\gamma(t) = X(u(t), v(t))$ where $(u(t), v(t))$ is a curve in $U \subset \mathbb{R}^2$ then

$$\kappa_N = [\dot\gamma]^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} [\dot\gamma] = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2.$$

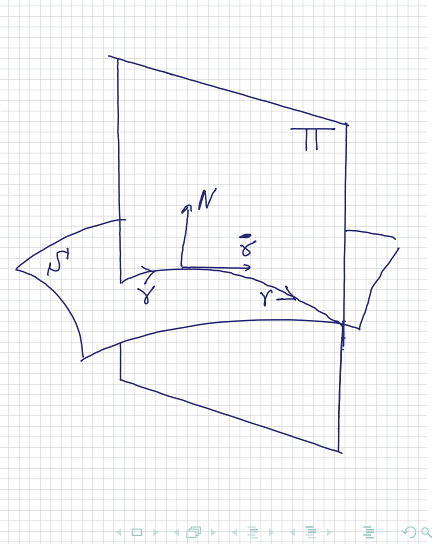
Theorem (Meusnier's theorem)

Any two curves that lie in a surface S and have a common tangent at some $p \in S$ have the same normal curvature at p.

Normal sections of a surface

Definition

A curve $\gamma : I \to \mathbb{R}^3$ in a surface S is a *normal section* if it is the curve of intersection of S with a plane Π perpendicular to $T_{\gamma(t)}S$ for every $t \in I$.



- γ is in both S and Π.
- $\frac{d^k\gamma}{dt^k} \in \Pi$ for all $k = 1, 2, \ldots$.
- Then $\dot{\gamma}, \ddot{\gamma} \in \Pi$.
- Unit speed: $\dot{\gamma} \perp \ddot{\gamma}$.
- Then $\ddot{\gamma}$ is parallel to $\mathbf{N}$.

Normal sections continued

- Let $\gamma(t)$ be a unit speed normal section of S with $\kappa \neq 0$.
- Let $\mathbf{n}$ be the principal normal to the curve: $\mathbf{n} = \frac{1}{\kappa}\ddot{\gamma}$.
- Recall $\kappa_N = \kappa \cos\psi$, $\kappa_g = \kappa \sin\psi$, where $\psi = \angle \mathbf{Nn}$.
- Since $\ddot{\gamma}$ is parallel to $\mathbf{N}$, then the principal normal $\mathbf{n}$ to the curve is parallel to $\mathbf{N}$ (the normal to S), so $\psi = 0$.
- Therefore, $\kappa_g = 0$ and $\kappa_N = \pm\kappa$ for a normal section with nonzero curvature. We may write its curvature as $\kappa = |\kappa_N| = |\langle\langle\dot{\gamma}, \dot{\gamma}\rangle\rangle|$.
- By Meusnier, all curves in S tangent to a normal section at $p \in S$ have the same κ_N at p.

Lecture 13: Parallel transport

Covariant derivative

What can "parallel" mean on an arbitrary surface?

- Vector field $\mathbf{v}$ in $\mathbb{R}^3$.
- Curve γ on a surface S in $\mathbb{R}^3$.
- $\dot{\mathbf{v}}$ is the derivative of $\mathbf{v}$ along γ.
- $\mathbf{N}$ is a unit normal field for S.
- $\dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N}$ is the component of $\dot{\mathbf{v}}$ tangent to S.

Definition (Covariant derivative along a curve)

Given the above, we write $\nabla_\gamma \mathbf{v} := \dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N}$ and call it the *covariant derivative* (sometimes called the *directional covariant derivative*, sometimes written $\nabla_{\dot{\gamma}} \mathbf{v}$) of $\mathbf{v}$ in the direction of $\dot{\gamma}$. It is the projection of $\dot{\mathbf{v}}$ into $T_{\gamma(t)}S$.

Parallel transport

Definition

If $\nabla_\gamma \mathbf{v} = 0$ along γ, we say that $\mathbf{v}$ is *parallel* (in physics: *parallel-transported* or *covariantly constant*) along γ.

Theorem

$\mathbf{v}$ *is parallel along* γ *if and only if* $\dot{\mathbf{v}} \perp T_{\gamma(t)}S$ *for all* t *in the domain of* γ*.*

Proof.

$\nabla_\gamma \mathbf{v} := \dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N} = 0$ if and only if $\dot{\mathbf{v}} = (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N}$.
Then $\dot{\mathbf{v}} \| \mathbf{N}$. But $\dot{\mathbf{v}} \| \mathbf{N}$ if and only if $\dot{\mathbf{v}} \perp T_{\gamma(t)}S$.
Conversely, if $\dot{\mathbf{v}} \perp T_{\gamma(t)}S$ for all t then $\dot{\mathbf{v}} \| \mathbf{N}$, and then necessarily $\dot{\mathbf{v}} = (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N}$, so $\nabla_\gamma \mathbf{v} := \dot{\mathbf{v}} - (\dot{\mathbf{v}} \cdot \mathbf{N})\,\mathbf{N} = 0$. □

Remark: If a vector field *in a plane* $\Pi \subset \mathbb{R}^3$ is parallel along a curve in Π, it is parallel in the usual sense of a translation isometry in the plane.

Christoffel symbols

Definition (Christoffel symbols)

Let $X : U \to \mathbb{R}^3$ be a coordinate patch and let $\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ be the 1FF of this patch. Note that $\det \mathcal{F}_I = EG - F^2$. The *Christoffel symbols* of the 1FF of this patch are

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)} \qquad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{GE_v - FG_u}{2(EG - F^2)} \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{EG_u - FE_v}{2(EG - F^2)}$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)} \qquad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}$$

Note: Christoffel symbols depend only on the 1FF, not the 2FF, on a surface patch.

When can tangent vector fields be parallel?

- Surface patch $X : U \to \mathbb{R}^3$. Then $\{X_u, X_v\}$ is a basis for T_pS.
- If $\mathbf{N}$ is normal to S at p then $\{X_u, X_v, \mathbf{N}\}$ is a basis for $\mathbb{R}^3$.
- Express X_{uu}, X_{uv}, X_{vv} in this basis:

$$\begin{aligned} X_{uu} &= a_1 X_u + a_2 X_v + a_3 \mathbf{N} \\ X_{uv} &= X_{vu} = b_1 X_u + b_2 X_v + b_3 \mathbf{N} \\ X_{vv} &= c_1 X_u + c_2 X_v + c_3 \mathbf{N}. \end{aligned}$$

  for coefficients $a_1, \dots, c_3$ which we will now find.
- To start, take dot products with $\mathbf{N}$:

$$\begin{aligned} \mathbf{N} \cdot X_{uu} &= a_3, \text{ but } L := \mathbf{N} \cdot X_{uu}, \text{ so } a_3 = L. \\ \mathbf{N} \cdot X_{uv} &= b_3, \text{ but } M := \mathbf{N} \cdot X_{uv}, \text{ so } b_3 = M. \\ \mathbf{N} \cdot X_{vv} &= c_3, \text{ but } N := \mathbf{N} \cdot X_{vv}, \text{ so } c_3 = N. \end{aligned}$$

...continued

- So now we have

$$\begin{aligned} X_{uu} &= a_1 X_u + a_2 X_v + L\mathbf{N} \\ X_{uv} &= X_{vu} = b_1 X_u + b_2 X_v + M\mathbf{N} \\ X_{vv} &= c_1 X_u + c_2 X_v + N\mathbf{N}. \end{aligned}$$

- Now take dot products with X_u:

$$\begin{aligned} X_u \cdot X_{uu} &= a_1 X_u \cdot X_u + a_2 X_u \cdot X_v = a_1 E + a_2 F \\ X_u \cdot X_{uv} &= b_1 X_u \cdot X_u + b_2 X_u \cdot X_v = b_1 E + b_2 F \\ X_u \cdot X_{vv} &= c_1 X_u \cdot X_u + c_2 X_u \cdot X_v = c_1 E + c_2 F. \end{aligned}$$

- Taking dot products with X_v yields

$$\begin{aligned} X_v \cdot X_{uu} &= a_1 X_v \cdot X_u + a_2 X_v \cdot X_v = a_1 F + a_2 G \\ X_v \cdot X_{uv} &= b_1 X_v \cdot X_u + b_2 X_v \cdot X_v = b_1 F + b_2 G \\ X_v \cdot X_{vv} &= c_1 X_v \cdot X_u + c_2 X_v \cdot X_v = c_1 F + c_2 G. \end{aligned}$$

- Need to simplify the left-hand sides.

...continued

- Consider the equation $X_u \cdot X_{uu} = a_1 E + a_2 F$.
  - Now $X_u \cdot X_{uu} = \frac{1}{2}\frac{\partial}{\partial u}(\|X_u\|^2) = \frac{1}{2}E_u$.
  - The above equation becomes $\frac{1}{2}E_u = a_1 E + a_2 F$.
- Consider the equation $X_v \cdot X_{uu} = a_1 F + a_2 G$.
  - $X_v \cdot X_{uu} = \frac{\partial}{\partial u}(X_v \cdot X_u) - \frac{1}{2}\frac{\partial}{\partial v}(X_u \cdot X_u) = F_u - \frac{1}{2}E_v$.
  - The above equation becomes $F_u - \frac{1}{2}E_v = a_1 F + a_2 G$.
- Solve for $a_1 = \frac{E_u G + E_v F - 2FF_u}{2(EG - F^2)}$, $a_2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)}$.
- But these are two of the Christoffel symbols: $a_1 = \Gamma^1_{11}$, $a_2 = \Gamma^2_{11}$.
- Continuing, we obtain that all the $a_1, \cdots, c_3$ are Christoffel symbols, and:

$$\begin{aligned}
X_{uu} &= a_1 X_u + a_2 X_v + L\mathbf{N} = \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\mathbf{N},\\
X_{uv} &= X_{vu} = b_1 X_u + b_2 X_v + M\mathbf{N} = \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{N},\\
X_{vv} &= c_1 X_u + c_2 X_v + N\mathbf{N} = \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{N}.
\end{aligned}$$

Gauss equations (first version)

Definition

The equations we just obtained are sometimes called the *Gauss equations*:

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\mathbf{N}, \\ X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{N}, \\ X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{N}. \end{aligned}$$

They provide a link between the 1FF (through the Christoffel symbols), the 2FF (last terms on right), and transport of vector fields (the basis vectors appearing on the left-hand sides).

We will use these to obtain related equations, also named for Gauss (and Codazzi and Mainardi) a few lectures from now.

Return to issue of parallel tangent fields

- Curve $\gamma(t) = X(u(t), v(t))$ in S, field $\mathbf{v}(t)$ along γ, tangent to S.

$$\begin{aligned}
\mathbf{v}(t) &= \alpha(t)X_u + \beta(t)X_v \in T_{\gamma(t)}S \\
\nabla_\gamma \mathbf{v} &= \dot{\alpha}X_u + \dot{\beta}X_v + \alpha\dot{X}_u^\perp + \beta\dot{X}_v^\perp \text{ where } \dot{X}_u^\perp := \dot{X}_u - \left(\dot{X}_u \cdot \mathbf{N}\right)\mathbf{N} \\
&= \dot{\alpha}X_u + \dot{\beta}X_v + \alpha\left(X_{uu}\dot{u} + X_{uv}\dot{v}\right)^\perp + \beta\left(X_{vu}\dot{u} + X_{vv}\dot{v}\right)^\perp \\
&= \dot{\alpha}X_u + \dot{\beta}X_v + \alpha\dot{u}\left(\Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v\right) \\
&\qquad + \left(\alpha\dot{v} + \beta\dot{u}\right)\left(\Gamma_{12}^1 X_u + \Gamma_{11}^2 X_v\right) + \beta\dot{v}\left(\Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v\right),
\end{aligned}$$

  using the Gauss equations in the last line.

- If $\mathbf{v}$ is parallel along γ, then $\dot{\mathbf{v}} \| \mathbf{N}$, so coefficients of X_u and X_v above must both vanish.

$$\begin{aligned}
0 &= \dot{\alpha} + \alpha\dot{u}\Gamma_{11}^1 + \left(\alpha\dot{v} + \beta\dot{u}\right)\Gamma_{12}^1 + \beta\dot{v}\Gamma_{22}^1 \\
0 &= \dot{\beta} + \alpha\dot{u}\Gamma_{11}^2 + \left(\alpha\dot{v} + \beta\dot{u}\right)\Gamma_{11}^2 + \beta\dot{v}\Gamma_{22}^2.
\end{aligned}$$

- These are the *equations of parallel transport*.

Equations of parallel transport

- We have proved that if a vector field $\mathbf{v} = \alpha(t)X_u + \beta(t)X_v$ tangent to S is parallel along a curve γ, then necessarily
$$0 = \dot{\alpha} + \alpha\dot{u}\Gamma^1_{11} + (\alpha\dot{v} + \beta\dot{u})\,\Gamma^1_{12} + \beta\dot{v}\Gamma^1_{22}$$
$$0 = \dot{\beta} + \alpha\dot{u}\Gamma^2_{11} + (\alpha\dot{v} + \beta\dot{u})\,\Gamma^2_{11} + \beta\dot{v}\Gamma^2_{22}.$$
- Conversely, this system of equations has form
$$\dot{\alpha}(t) = f(\alpha, \beta, t)$$
$$\dot{\beta}(t) = g(\alpha, \beta, t).$$

  for smooth functions f, g. From ODE theory, there is always a unique solution $(\alpha(t), \beta(t))$ (from which we can then construct $\mathbf{v}$) on some open interval containing t_0, given initial values $\alpha_0 = \alpha(t_0)$, $\beta_0 = \beta(t_0)$. This proves:

Theorem

Let γ be a curve on S. Let $\mathbf{v}_0 \in T_pS$, where $p = \gamma(t_0)$. Then there is exactly one vector field in $T_{\gamma(t)}S$ that is parallel along γ and equal to $\mathbf{v}_0$ at p.

The parallel transport map

- Let $p, q \in S$ be two points along curve γ in S, where $\gamma(t_0) = p$ and $\gamma(t_1) = q$.
- Define a map $\Pi_\gamma^{p,q} : T_pS \to T_qS$ as follows:
  - Given $\mathbf{v}_0 \in T_pS$, say $\mathbf{v}(t)$ is the unique parallel vector field along $\gamma : [t_0, t_1] \to \mathbb{R}^3$ with $\mathbf{v}(t_0) = v_0$.
  - Then $\Pi_\gamma^{p,q} v_0 := v_1$ where $v_1 := v(t_1)$.

Theorem

- $\Pi_\gamma^{p,q}$ *is linear.*
- $\Pi_\gamma^{p,q}$ *is an isometry:*

$$\langle \mathbf{v}_0, \mathbf{w}_0 \rangle_p = \langle \mathbf{v}_1, \mathbf{w}_1 \rangle_q \text{ for } \mathbf{v_1} = \Pi_\gamma^{p,q}\mathbf{v}_0, \mathbf{w_1} = \Pi_\gamma^{p,q}\mathbf{w}_0.$$

Proof: text pp 175–176.

Example: Sphere (minus the poles)

- $\theta =$ latitude, $\varphi =$ azimuth (longitude).
- First patch $U_1 = \left\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}, 0 < \varphi < 2\pi\right\}$.
- Second patch $U_2 = \left\{-\frac{\pi}{2} < \theta < \frac{\pi}{2}, -\pi < \varphi < \pi\right\}$.
- $X(\theta, \varphi) = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$ for both patches.
- $\mathcal{F}_I = d\theta^2 + \cos^2\theta d\varphi^2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & \cos^2\theta \end{array}\right]$.
- $\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = 0$, $\Gamma^2_{12} = \Gamma^2_{21} = -\tan\theta$, $\Gamma^1_{22} = -\sin\theta\cos\theta$.
- Along any constant-latitude circle $\theta = \theta_0$, $\varphi = t$, a parallel vector field $\mathbf{v}$ obeys

$$\begin{aligned} \mathbf{v} &= \alpha X_\theta + \beta X_\varphi \\ \dot{\alpha} &= -\beta\sin\theta_0\cos\theta_0 \\ \dot{\beta} &= \alpha\tan\theta_0 \end{aligned}$$

- Equator: $\theta_0 = 0$, so $\alpha(\varphi) = \alpha_0$, $\beta(\varphi) = \beta_0$, $\mathbf{v}(\varphi) = \alpha_0 X_\theta + \beta_0 X_\varphi$.

Sphere example continued

- Last page: $\mathbf{v} = \alpha X_\theta + \beta X_\varphi$, where $\dot{\alpha} = -\beta \sin\theta_0 \cos\theta_0$ and $\dot{\beta} = \alpha \tan\theta_0$.
- If $\theta_0 \neq 0$, differentiate middle equation again and use bottom equation:

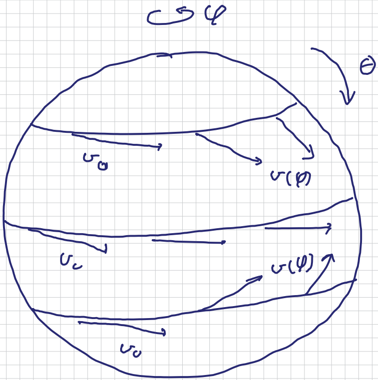
$$\begin{aligned} &\implies \ddot{\alpha} = -\alpha \sin^2\theta_0 \\ &\implies \ddot{\alpha} + (\sin^2\theta_0)\alpha = 0 \\ &\implies \alpha(\varphi) = A\cos(\varphi\sin\theta_0) + B\sin(\varphi\sin\theta_0). \end{aligned}$$

- Then $\beta(\varphi) = A\frac{\sin(\varphi\sin\theta_0)}{\cos\theta_0} - B\frac{\cos(\varphi\sin\theta_0)}{\cos\theta_0}$.
- Special case: $\mathbf{v}(0) = X_\varphi$ so $\alpha(0) = 0$ and $\beta(0) = 1$. Then $A = 0$, $B = \cos\theta_0$, so $\alpha(\varphi) = -(\cos\theta_0)\sin(\varphi\sin\theta_0)$ and $\beta(\varphi) = \cos(\varphi\sin\theta_0)$.

$$\implies \mathbf{v}(\varphi) = -(\cos\theta_0)\sin(\varphi\sin\theta_0)X_\theta + \cos(\varphi\sin\theta_0)X_\varphi.$$

- Parallel vector field initially tangent to γ cannot remain so. (What happens after one complete cycle around γ?)

Parallel vector field on sphere



Lecture 14: Gaussian and mean curvatures

Self-adjointness of W

Definition (Adjoint)

Say A and B are operators on a vector space V and $\langle\cdot,\cdot\rangle$ is an inner product for V. If $\langle \mathbf{v}, A(\mathbf{w})\rangle = \langle B(\mathbf{v}), \mathbf{w}\rangle$ for all $\mathbf{v}, \mathbf{w} \in V$ then we say that B is the *adjoint* of A with respect to the inner product $\langle\cdot,\cdot\rangle$.

- Recall: The 2FF can be written as a symmetric matirx, so the Weingarten map is *self-adjoint* wrt the inner product defined by the 1FF:

$$\langle W(\mathbf{v}), \mathbf{w}\rangle = \langle\langle \mathbf{v}, \mathbf{w}\rangle\rangle = \langle\langle \mathbf{w}, \mathbf{v}\rangle\rangle = \langle W(\mathbf{w}), \mathbf{v}\rangle = \langle \mathbf{v}, W(\mathbf{w})\rangle.$$

- $W = W_{p,S} : T_pS \to T_pS$ is a linear operator.
- Self-adjoint linear operators have *real* eigenvalues, so

$$\begin{aligned} W(\mathbf{t}_1) &= \kappa_1 \mathbf{t}_1, \\ W(\mathbf{t}_2) &= \kappa_2 \mathbf{t}_2, \end{aligned}$$

with $\kappa_1, \kappa_2 \in \mathbb{R}$ and $\|\mathbf{t}_1\| \neq 0$, $\|\mathbf{t}_2\| \neq 0$.

Eigenvectors of self-adjoint operators produce orthonormal bases

- Last slide: $W(\mathbf{t}_1) = \kappa_1 \mathbf{t}_1$, $W(\mathbf{t}_2) = \kappa_2 \mathbf{t}_2$, $\kappa_1, \kappa_2 \in \mathbb{R}$.
- If $\kappa_1 \neq \kappa_2$ then

$$\begin{aligned} & \langle W(\mathbf{t}_1), \mathbf{t}_2 \rangle = \langle \mathbf{t}_1, W(\mathbf{t}_2) \rangle \text{ since } W \text{ is self-adjoint} \\ \implies & \kappa_1 \langle \mathbf{t}_1, \mathbf{t}_2 \rangle = \kappa_2 \langle \mathbf{t}_1, \mathbf{t}_2 \rangle \\ \implies & (\kappa_2 - \kappa_1) \langle \mathbf{t}_1, \mathbf{t}_2 \rangle = 0. \end{aligned}$$

  and since $\kappa_2 \neq \kappa_1$ then we must have $\mathbf{t}_1 \perp \mathbf{t}_2$.
- Eigenvectors belonging to *distinct* eigenvalues are orthogonal; normalize them to obtain an orthonormal basis (ONB) $\{\mathbf{t}_1, \mathbf{t}_2\}$ for $\mathbb{R}^2$.
- If $\kappa_1 = \kappa_2$, the eigenspace is 2-dimensional, and from it we can choose eigenvectors that form an ONB $\{\mathbf{t}_1, \mathbf{t}_2\}$.
- We will always label eigenvectors so that $\{\mathbf{t}_1, \mathbf{t}_2\}$ is right-handed.

Principal curvatures

Definition (Principal curvatures)

The eigenvalues κ_1 and κ_2 of W are the *principal curvatures* of surface S. The corresponding eigenvectors are the *principal vectors* or *principal directions*.

- We can always find an orthonormal basis for T_pS whose elements are principal vectors.
- Points at which $\kappa_1 = \kappa_2$ are called *umbilics*. At umbilics, the eigenspace is 2-dimensional, so it's all of T_pS, and then:

$$W_{p,S}(\mathbf{t}) = \kappa \mathbf{t}$$

  for all $\mathbf{t} \in T_p, S$, where we write $\kappa = \kappa_1 = \kappa_2$.
- Therefore at umbilics $W = \kappa\,\mathrm{id}$ (id is the identity map $\mathrm{id}(\mathbf{t}) = \mathbf{t}$).

Mean and Gauss curvatures

In an eigenvector basis, the matrix $\mathcal{W}$ for the Weingarten map is

$$\mathcal{W} = \begin{bmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{bmatrix}$$

Definition (Mean curvature)

The mean curvature H of a surface is one-half the trace of $\mathcal{W}$:

$$H := \frac{1}{2}\,\mathrm{tr}(\mathcal{W}) = \frac{1}{2}\left(\kappa_1 + \kappa_2\right).$$

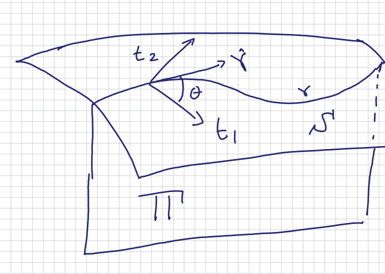
Definition (Gauss curvature)

The Gauss curvature $K = K_G$ is the determinant of $\mathcal{W}$:

$$K = K_G = \det(\mathcal{W}) = \kappa_1\kappa_2.$$

Interpretation

- Say $\{\mathbf{t}_1, \mathbf{t}_2\}$ is an ONB for T_p at $p \in S$, consisting of eigenvectors of W.
- Say Π is a plane through p and containing the normal $\mathbf{N}$ to S at p.
- Then the (unit speed) curve of intersection γ of Π and S is a normal section.
- Say $\dot{\gamma}$ makes angle θ with $\mathbf{t}_1$, so $\dot{\gamma} = \cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2$.



Interpretation continued

- Since γ is a normal section, $\kappa = \kappa_N = \langle\langle \dot{\gamma}, \dot{\gamma} \rangle\rangle_{p,S}$.
- From last slide, $\dot{\gamma} = \cos\theta \mathbf{t}_1 + \sin\theta \mathbf{t}_2$.
- Then $\kappa_N = \cos^2\theta \langle\langle \mathbf{t}_1, \mathbf{t}_1 \rangle\rangle + 2\cos\theta\sin\theta\langle\langle \mathbf{t}_1, \mathbf{t}_2 \rangle\rangle + \sin^2\theta\langle\langle \mathbf{t}_2, \mathbf{t}_2 \rangle\rangle$.
- Moreover, $\langle\langle \mathbf{t}_i, \mathbf{t}_j \rangle\rangle = \langle W(\mathbf{t}_i), \mathbf{t}_j \rangle = \kappa_i \langle \mathbf{t}_i, \mathbf{t}_j \rangle = \begin{cases} \kappa_i, & i = j, \\ 0, & i \neq j. \end{cases}$
- Combine last two lines:

$$\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.$$

- Any curve tangent to γ at p will have same κ_N.

Theorem

Theorem

κ_1 and κ_2 are the extreme values of the normal curvature κ_N among all curves at p. The max and min values occur for normal sections in orthogonal planes.

Proof:

- Meusnier's theorem: two curves through p have same κ_N if they have same tangent at p, so it suffices to extremize over unit speed normal section curves.
- From last slide, for these curves $\kappa_N = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta$.
- If $\kappa_1 = \kappa_2$, then $\kappa_N = \kappa_1\left(\cos^2\theta + \sin^2\theta\right) = \kappa_1 = \kappa_2$ and κ_N is constant with respect to θ, hence constant over all curves through p.
- If $\kappa_1 \neq \kappa_2$, then extremize:
- $0 = \frac{d}{d\theta}\kappa_N = 2\left(\kappa_2 - \kappa_1\right)\sin\theta\cos\theta = \left(\kappa_2 - \kappa_1\right)\sin(2\theta)$, so $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ at extrema.
- If $\theta = 0, \pi, 2\pi$, then $\kappa_N = \kappa_1$. If $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$, then $\kappa_N = \kappa_2$.

The matrix of $W_{p,X}$

- Coordinate patch $X : U \to \mathbb{R}^3$ for $S \ni p$, $(u, v) \in U$.
- Basis $\{X_u, X_v\}$.
- Matrix for 1FF: $\mathcal{F}_I = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$.
- Matrix for 2FF: $\mathcal{F}_{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$.
- Write matrix for W as $\mathcal{W} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ for unknowns a, b, c, d.
- Use $\langle W(X_u), X_u \rangle = \langle\langle X_u, X_u \rangle\rangle$. In matrix form, this is

$$\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} E \\ F \end{bmatrix} = L$$

$$\implies aE + bF = L.$$

Matrix for $W_{p,X}$ continued

- Last slide: We used $\langle W(X_u), X_u\rangle = \langle\langle X_u, X_u\rangle\rangle$ to get $aE + bF = L$.
- Next use $\langle W(X_u), X_v\rangle = \langle\langle X_u, X_v\rangle\rangle$:

$$\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)^T \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} L & M \\ M & N \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} = M$$

$$\Longrightarrow aF + bG = M.$$

- Likewise $\langle W(X_v), X_u\rangle = \langle\langle X_v, X_u\rangle\rangle$ yields $cE + dF = M$.
- $\langle W(X_v), X_v\rangle = \langle\langle X_v, X_v\rangle\rangle$ yields $cF + dG = N$.

Matrix for $W_{p,X}$ ... endgame

- For the unknown elements a, b, c, d of $\mathcal{W}$ we have $aE + bF = L$, $aF + bG = M$, $cE + dF = M$, $cF + dG = N$. Can write these four as the matrix equation

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}.$$

- This is $\mathcal{F}_I \mathcal{W} = \mathcal{F}_{II}$.
- Then for the matrix of $W_{p,X}$ we get

$$\begin{aligned} \mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II} &= \frac{1}{(EG - F^2)} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} L & M \\ M & N \end{bmatrix} \\ &= \frac{1}{(EG - F^2)} \begin{bmatrix} GL - FM & GM - FN \\ EM - FL & EN - FM \end{bmatrix}. \end{aligned}$$

- Mean curvature: $H = \frac{1}{2} \operatorname{tr} \mathcal{W} = \frac{GL+EN-2FM}{2(EG-F^2)}$.
- Gauss curvature: $K_G = \det \mathcal{W} = \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I} = \frac{LN-M^2}{EG-F^2}$.
- Can also extract formulas for κ_1, κ_2 in terms of $E, \dots, N$.

Example: Surface of revolution

- Unit speed profile curve in xz-plane: $x = f(u)$, $y = 0$, $z = g(u)$, $\dot{f}^2(u) + \dot{g}^2(u) = 1$.
- $X(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$, $f(u) \geq 0$, $\dot{g}(u) \neq 0$.
- $X_u = \left(\dot{f}(u) \cos v, \dot{f}(u) \sin v, \dot{g}(u)\right)$.
- $X_v = (-f(u) \sin v, f(u) \cos v, 0)$.
- $\mathcal{F}_I = \begin{bmatrix} \|X_u\|^2 & X_u \cdot X_v \\ X_u \cdot X_v & \|X_v\|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & f^2(u) \end{bmatrix}$.

Surface of revolution continued

- $X_u \times X_v = \Big(-f(u)\dot{g}(u)\cos v, -f(u)\dot{g}(u)\sin v, f(u)\dot{f}(u) \Big)$.
- $\|X_u \times X_v\| = \sqrt{f^2(u)\dot{g}^2(u) + f^2(u)\dot{f}^2(u)} = f(u)$.
- $\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \Big(-\dot{g}(u)\cos v, -\dot{g}(u)\sin v, \dot{f}(u) \Big)$.
- $X_{uu} = \Big(\ddot{f}(u)\cos v, \ddot{f}(u)\sin v, \ddot{g}(u) \Big)$.
- $X_{uv} = X_{vu} = \Big(-\dot{f}(u)\sin v, \dot{f}(u)\cos v, 0 \Big)$.
- $X_{vv} = (-f(u)\cos v, -f(u)\sin v, 0)$.
- $\mathcal{F}_{II} = \begin{bmatrix} \mathbf{N}\cdot X_{uu} & \mathbf{N}\cdot X_{uv} \\ \mathbf{N}\cdot X_{vu} & \mathbf{N}\cdot X_{vv} \end{bmatrix} = \begin{bmatrix} \dot{f}\ddot{g} - \ddot{f}\dot{g} & 0 \\ 0 & f\dot{g} \end{bmatrix}$.

Surface of revolution continued

- $\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II} = \begin{bmatrix} 1 & 0 \\ 0 & 1/f^2 \end{bmatrix} \begin{bmatrix} \dot{f}\ddot{g} - \ddot{f}\dot{g} & 0 \\ 0 & f\dot{g} \end{bmatrix} = \begin{bmatrix} \dot{f}\ddot{g} - \ddot{f}\dot{g} & 0 \\ 0 & \dot{g}/f \end{bmatrix}$.
- $H = \frac{1}{2}\operatorname{tr}\mathcal{W} = \frac{1}{2}\left[\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f}\right]$.
- $K_G = \det\mathcal{W} = \frac{\dot{g}}{f}\left[\dot{f}\ddot{g} - \ddot{f}\dot{g}\right]$.
- Special case: Sphere of radius $a > 0$:
  - Unit speed profile curve $\gamma(u) = \left(a\cos\frac{u}{a}, 0, a\sin\frac{u}{a}\right)$.
  - surface $X(u, v) = \left(a\cos\frac{u}{a}\cos v, a\cos\frac{u}{a}\sin v, a\sin\frac{u}{a}\right)$.
  - $f(u) = a\cos\frac{u}{a} \implies \dot{f}(u) = -\sin\frac{u}{a} \implies \ddot{f}(u) = -\frac{1}{a}\cos\frac{u}{a}$.
  - $g(u) = a\sin\frac{u}{a} \implies \dot{g}(u) = \cos\frac{u}{a} \implies \ddot{g}(u) = -\frac{1}{a}\sin\frac{u}{a}$.
  - Then $\dot{f}\ddot{g} - \ddot{f}\dot{g} = \frac{1}{a}\sin^2\frac{u}{a} + \frac{1}{a}\cos^2\frac{u}{a} = \frac{1}{a}$ and $\frac{\dot{g}}{f} = \frac{1}{a}$.
  - Then $H = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{a}\right) = \frac{1}{a}$.
  - And $K_G = \frac{1}{a^2}$.
  - *Important*: Notice the dimensions. H (and κ_1, κ_2) have dimension $[\text{distance}]^{-1}$. K has dimension $[\text{distance}]^{-2}$.

Standard tori

Definition (Standard torus in $\mathbb{R}^3$)

A standard torus in $\mathbb{R}^3$ is any torus in the family of surfaces of revolution obtained by revolving the profile curves

$$\gamma(u) = \left(a + b\cos\frac{u}{b}, 0, b\sin\frac{u}{b}\right),\ a \geq b,\ u \in [0, 2\pi),$$

about the z-axis (the vertical axis).

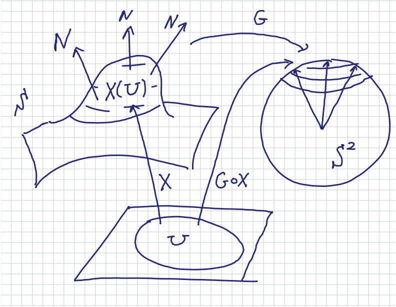
Exercise: For a standard torus $\mathcal{T}$:

- Find κ_1, κ_2, H, K_G.
- Compute the *Willmore energy* $W(a/b) = \int_{\mathcal{T}} H^2 dA$.
- The Willmore energy of a standard torus is $W(z)$ is a function of the single variable $z = a/b$. Find z such that $W(z)$ is a minimum. Standard tori with a/b given by this value are called Willmore tori.
- For more information, google "Willmore conjecture".

Lecture 15: Principal curvatures

Surface area and Gauss curvature

- Gauss map: $S : S \to \mathbb{S}^2 : p \mapsto \mathbf{N}_p$.
- Disk: $U = \{u^2 + v^2 \leq \delta^2\} \subset \mathbb{R}^2$.
- Image of disk: $R = X(U) \subset S$.
- Area of R: $A(R) = \int_U \|X_u \times X_v\| \, dudv$, where $\{X_u, X_v\}$ is a basis for T_pS.
- $G \circ X : U \to \mathbb{S}^2$.
- $(G \circ X)(u, v) = N_p$.
- Area of $G(R)$ is $\int_U \|(G \circ X)_u \times (G \circ X)_v\| \, dudv$.



Compare areas

- Compare $A(R)$ and $A(G(R))$.

$$\frac{A_{G\circ X}(G(R))}{A_X(R)} = \frac{\int_U \|(G\circ X)_u \times (G\circ X)_v\|\, dudv}{\int_U \|X_u \times X_v\|\, dudv}$$

- To begin, recall (Ch 7)

$$\begin{aligned}
W(X_u)(u_0, v_0) &= -\frac{d}{du}\Big|_{u=u_0} G(X(u, v_0)) = -\mathbf{N}_u(u_0, v_0) \\
W(X_v)(u_0, v_0) &= -\frac{d}{dv}\Big|_{v=v_0} G(X(u_0, v)) = -\mathbf{N}_v(u_0, v_0) \\
\implies \mathbf{N}_u \times \mathbf{N}_v &= W(X_u) \times W(X_v) = (aX_u + cX_v) \times (bX_u + dX_v)
\end{aligned}$$

using $\mathcal{W} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ in $\{X_u, X_v\}$ basis. Then

$(G\circ X)_u \times (G\circ X)_v = (ad - bc)X_u \times X_v = (\det \mathcal{W})X_u \times X_v = K_G X_u \times X_v$.

Compare areas...continued

- Then

$$\begin{aligned}\frac{A_{G\circ X}(G(R))}{A_X(R)} &= \frac{\int_U \|(G\circ X)_u \times (G\circ X)_v\|\, dudv}{\int_U \|X_u \times X_v\|\, dudv}\\ &= \frac{\int_U |K_G|\, \|X_u \times X_v\|\, dudv}{\int_U \|X_u \times X_v\|\, dudv}\\ &= \frac{\int_U |K_G|\, dA_X}{\int_U dA_X}.\end{aligned}$$

- Take radius δ of disk U to be arbitrarily small. Then $K_G \to K_G(u_0, v_0) = K_0 = const$ and so

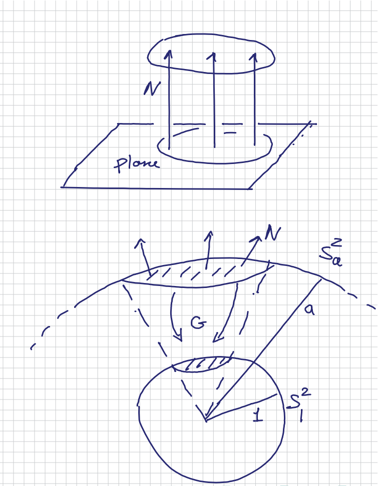
$$\frac{A_{G\circ X}(G(R))}{A_X(R)} = \frac{\int_U |K_G|\, dA_X}{\int_U dA_X} \to |K_0|\,.$$

Simple Gauss curvature calculations

- Plane: Normals are parallel in $\mathbb{R}^3$ so $K_G = 0$.
- Sphere $\mathbb{S}^2_a$ of radius $a > 0$. Then $\mathbf{N} = G = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\mathbf{r}}{a}$. Then $G(\mathbb{S}^2_a) = \mathbb{S}^2_1$, so

$$\frac{A(G(\mathbb{S}^2_a))}{A(\mathbb{S}^2_a)} = \frac{A(\mathbb{S}^2_1)}{A(\mathbb{S}^2_a)} = \frac{4\pi}{4\pi a^2} = \frac{1}{a^2}$$

$$\implies |K_G| = 1/a^2.$$



Umbilics

Theorem

If every point of surface S is an umbilic, then S is an open subset of a plane or a sphere.

Proof:

- Umbilics are points p with $\kappa_1 = \kappa_2 =: \kappa$. Then $W(\mathbf{t}) = \kappa\mathbf{t}$ for every $\mathbf{t} \in T_p$.
- At an umbilic then $W(X_u) = \kappa X_u$, $W(X_v) = \kappa X_v$.
- But $W(X_u) = -\frac{d}{du}\Big|_{u=u_0} G(X(u, v_0)) = -\mathbf{N}_u$. Likewise, $W(X_v) = -\mathbf{N}_v$.
- Conclude that $\kappa X_u = -\mathbf{N}_u$, $\kappa X_v = -\mathbf{N}_v$ at umbilic.
- If every point of S is umbilic, these equations hold everywhere. Hence we can differentiate them.
- Then $(\kappa X_u)_v = -\mathbf{N}_{uv}$ and $(\kappa X_v)_u = -\mathbf{N}_{vu}$.
- These equations have same right-hand side, so the left-hand sides equal. Expanding and simplifying, then

$$\kappa_v X_u = \kappa_u X_v$$

Proof continued

- Since $\{X_u, X_v\}$ is a linearly independent set, $\kappa_v X_u = \kappa_u X_v$ can hold only if $\kappa_u = 0 = \kappa_v$ everywhere.
- Thus κ is constant.
- Say $\kappa = 0$.
  - We had $\kappa X_u = -\mathbf{N}_u$, $\kappa X_v = -\mathbf{N}_v$, so $\mathbf{N}_u = 0 = \mathbf{N}_v$.
  - Then $\mathbf{N}$ is constant, and S must be (an open subset of) a plane.
- Say κ is a nonzero constant.
  - We still have $\kappa X_u = -\mathbf{N}_u$ and $\kappa X_v = -\mathbf{N}_v$.
  - Then $\kappa X = -\mathbf{N} + \mathbf{a}$, for a constant vector $\mathbf{a}$.
  - Then $-\frac{1}{\kappa}\mathbf{N} = X - \frac{1}{\kappa}\mathbf{a}$. Because $\|\mathbf{N}\| = 1$ then
    $$\frac{1}{\kappa^2} = \left\| X - \frac{1}{\kappa}\mathbf{a} \right\|^2 .$$
  - This says that $\frac{1}{\kappa^2} = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$, where $\frac{\mathbf{a}}{\kappa} = (x_0, y_0, z_0)$. It's the equation of a sphere.

The 2FF of a graph

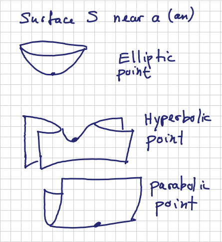
- $z = f(x, y)$ defines a graphical surface $S \subset \mathbb{R}^3$.
- Parametrize: $x = u$, $y = v$, $z = f(u, v)$.
- One patch covers a graph: $X : U \to \mathbb{R}^3 : (u, v) \mapsto (u, v, f(u, v))$.
- Basis vectors for T_pS: $X_u = (1, 0, f_u)$, $X_v = (0, 1, f_v)$.
- 1FF: $\mathcal{F}_I = \begin{bmatrix} \|X_u\|^2 & X_u \cdot X_v \\ X_u \cdot X_v & \|X_v\|^2 \end{bmatrix} = \begin{bmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{bmatrix}$.
- $X_u \times X_v = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = (-f_u, -f_v, 1)$.
- $\mathbf{N} = \frac{X_u \times X_v}{\|X_u \times X_v\|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$.

The 2FF of a graph continued

- $X_{uu} = (0, 0, f_{uu})$, $X_{uv} = X_{vu} = (0, 0, f_{uv})$, $X_{vv} = (0, 0, f_{vv})$.
- From last slide: $\mathbf{N} = \frac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}} = \frac{(-f_u, -f_v, 1)}{\sqrt{1+|\nabla f|^2}}$.
- 2FF: $\mathcal{F}_{II} = \begin{bmatrix} \mathbf{N} \cdot X_{uu} & \mathbf{N} \cdot X_{uv} \\ \mathbf{N} \cdot X_{vu} & \mathbf{N} \cdot X_{vv} \end{bmatrix} = \frac{1}{\sqrt{1+|\nabla f|^2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix}$.
- Special case of $z = f(x, y) = au^2 + bv^2$:
  - $\mathcal{F}_{II} = \frac{2}{\sqrt{1+4a^2u^2+4b^2v^2}} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.
  - At a critical point $\nabla f = (2au, 2bv) = (0, 0)$ then $\mathcal{F}_{II} = 2 \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\mathcal{F}_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II} = 2 \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ at the critical point $(0, 0)$ of $f(x, y) = au^2 + bv^2$.
  - So $z = f(u, v) = \frac{1}{2}\left(\hat{\kappa}_1 u^2 + \hat{\kappa}_2 v^2\right)$, for $\hat{\kappa}_1, \hat{\kappa}_2$ the principal curvatures at $(0, 0)$, and $H(0, 0) = a + b = \hat{\kappa}_1 + \hat{\kappa}_2$ and $K_G(0, 0) = 4ab = 4\hat{\kappa}_1\hat{\kappa}_2$.

$z = f(u,v) = \frac{1}{2}\left(\hat{\kappa}_1 u^2 + \hat{\kappa}_2 v^2\right)$

1. *Elliptic point*: $K_G > 0$, and either κ_1, κ_2 both positive, or both negative. S resembles elliptic parabola.
2. *Hyperbolic point*: $K_G < 0$, and κ_1, κ_2 have opposite signs. S resembles hyperboloid.
3. *Parabolic point*: $K_G = 0$ but only one of κ_1, κ_2 is zero. S resembles parabolic cylinder.
4. *Planar point*: $K_G = 0$, $\kappa_1 = \kappa_2 = 0$. S doesn't necessarily resemble a plane (see text p 194).



Compact $K_G \leq 0$ surfaces do not embed in $\mathbb{R}^3$

The following *obstruction* prevents flat tori and compact hyperbolic surfaces from globally isometrically embedding in $\mathbb{R}^3$.

Theorem

If $S \in \mathbb{R}^3$ is a compact surface it has a point where $K_G > 0$.

Proof:

- Define $F : \mathbb{R}^3 \to \mathbb{R} : \mathbf{v} \mapsto F(\mathbf{v}) = \|\mathbf{v}\|^2$.
- Let $S =$ compact surface, $O =$ origin of $\mathbb{R}^3$.
- Let $f(P) = F(\vec{OP})$ for $\vec{OP}$ the vector from O to $P \in S$.
- Maximum principle: Every continuous function with compact domain has a maximum.
- Then f has a maximum. Call the maximum a^2, where P is the furthest point on S from O and $a = \|\vec{OP}\|$

Proof continued

- Let γ be a unit speed curve on S passing through P, with $\gamma(0) = P$.
- Then $f(\gamma(t))$ has a maximum $f(\gamma(0)) = f(P) = a^2$ at $t = 0$.
- Therefore $\frac{d}{dt}\big|_{t=0} f(\gamma(t)) = 0$.
- Second derivative test: $\frac{d^2}{dt^2}\big|_{t=0} f(\gamma(t)) \leq 0$.
- From $\frac{d}{dt}\big|_{t=0} f(\gamma(t)) = 0$ and $f(\gamma(t)) = \|\gamma(t)\|^2$, we have

$$0 = 2\gamma(0) \cdot \dot{\gamma}(0) \tag{1}$$

  (so $\gamma(0) = \vec{OP} \perp T_P S$; therefore $\gamma(0) = \vec{OP} \| \mathbf{N}$).

- From $\frac{d^2}{dt^2}\big|_{t=0} f(\gamma(t)) \leq 0$ and $f(\gamma(t)) = \|\gamma(t)\|^2$, we have

$$0 \geq 2\gamma(0) \cdot \ddot{\gamma}(0) + 2\dot{\gamma}(0) \cdot \dot{\gamma}(0) = 2\left(\gamma(0) \cdot \ddot{\gamma}(0) + 1\right). \tag{2}$$

Proof continued

- From (1) on last slide, $\gamma(0) \perp \dot{\gamma}(0)$. Thus $\vec{OP} = \gamma(0) \perp T_P S$, so $\mathbf{N} = \frac{\vec{OP}}{\|\vec{OP}\|} = \frac{1}{a}\vec{OP}$ is normal to S at P.
- Recall: For any unit speed curve in S: $\ddot{\gamma} = \kappa_N \mathbf{N} + \kappa_g \mathbf{N} \times \dot{\gamma}$.
- Then $\kappa_N(0) = \mathbf{N} \cdot \ddot{\gamma}(0) = \frac{1}{a}\vec{OP} \cdot \ddot{\gamma}(0) = \frac{1}{a}\gamma(0) \cdot \ddot{\gamma}(0) \leq -\frac{1}{a}$ by (2) of last slide.
- Then $\kappa_N(0) \leq -1/a$.
- This must hold for all unit speed curves in S through P, so the maximum of κ_N over all such curves at P is $\leq -1/a$.
- Since the maximum and minimum of κ_N through a fixed point are principal curvatures, we have $\kappa_1 \leq -1/a$ and $\kappa_2 \leq -1/a$.
- Therefore $K_G = \kappa_1 \kappa_2 \geq 1/a^2 > 0$ at $P \in S$. QED.

Lecture 16: Geodesics on surfaces

Geodesics and minimal curves

Definition

A *geodesic* in S is a curve γ such that $\ddot{\gamma}$ is perpendicular to the tangent plane $T_{\gamma(t)}S$ (including possibly $\ddot{\gamma} = \mathbf{0}$) for each t.

If γ is a geodesic in S and $\mathbf{N}$ is normal to S, then $\ddot{\gamma} \| \mathbf{N}$ (including possibly $\ddot{\gamma} = \mathbf{0}$).

Properties:

- Geodesics have constant speed.
  Proof: $\frac{d}{dt}(\dot{\gamma} \cdot \dot{\gamma}) = 2\dot{\gamma} \cdot \ddot{\gamma} = 0$ since $\ddot{\gamma} \perp T_{\gamma(t)}S$, so $\dot{\gamma} \cdot \dot{\gamma} = const$.
- A unit speed curve γ is geodesic if and only if it has zero geodesic curvature $\kappa_g = 0$.
  Proof: Recall $\kappa_g := \ddot{\gamma} \cdot (\mathbf{N} \times \dot{\gamma})$. First, if γ is geodesic then either $\ddot{\gamma} = \mathbf{0}$ or $\ddot{\gamma} \| \mathbf{N}$; either way we see that $\kappa_g = 0$. Conversely, if $\kappa_g = 0$ then either $\ddot{\gamma} = \mathbf{0}$ of $\ddot{\gamma} \perp \mathbf{N} \times \dot{\gamma}$, and then $\ddot{\gamma} \in \text{Span}\{\mathbf{N}, \dot{\gamma}\}$. Since $\dot{\gamma} \cdot \ddot{\gamma} = 0$, then $\ddot{\gamma} \| \mathbf{N}$. But then γ is geodesic.

Properties continued

- Recall a vector $\mathbf{v}$ is *parallel* along γ iff $\dot{\mathbf{v}} \perp T_{\gamma(t)}S$. Letting $\mathbf{v} = \dot{\gamma}$ then:

$$\boxed{\text{Geodesics in } S \text{ parallel-transport their own tangent vectors.}}$$

$$\nabla_{\gamma}\dot{\gamma} = \ddot{\gamma} - (\ddot{\gamma} \cdot \mathbf{N})\,\mathbf{N} = \mathbf{0}$$

- If a (segment of a) straight line in $\mathbb{R}^3$ lies on a surface S, it's a geodesic of S. Proof: Can parametrize line as $\gamma(t) = \mathbf{a}t + \mathbf{b}$ (unit speed: $\|\mathbf{a}\| = 1$). Then $\ddot{\gamma}(t) = 0$.
- Any normal section of S, parametrized by arclength, is a geodesic. (Recall that normal sections are curves of intersection of S with a plane that contains the normal to S. As a special case, great circles are geodesics.)

The geodesic equations: set-up

- Patch $X : U \to \mathbb{R}^3 : (u, v) \mapsto X(u, v)$.
- Let $(u(t), v(t))$ be a curve in U.
- Then $\gamma(t) = X(u(t), v(t))$ is a curve in S, unit speed (reparametrize if necessary).
- Tangent: $\dot{\gamma}(t) = \frac{\partial X}{\partial u}\frac{du}{dt} + \frac{\partial X}{\partial v}\frac{dv}{dt} = X_u\dot{u}(t) + X_v\dot{v}(t)$.
- γ is geodesic, so $\ddot{\gamma} || \mathbf{N}$ for $\mathbf{N}$ normal to S.
- Then $\ddot{\gamma} \cdot X_u = 0$, $\ddot{\gamma} \cdot X_v = 0$.
- Equivalently, $\ddot{\gamma} - (\ddot{\gamma} \cdot \mathbf{N})\mathbf{N} = \mathbf{0}$.
- In other words, $\nabla_{\gamma}\dot{\gamma} = \mathbf{0}$.

If $\ddot\gamma \| \mathbf{N}$ then $\ddot\gamma \cdot X_u = 0$, $\ddot\gamma \cdot X_v = 0$.

- Consider the equation $\ddot\gamma \cdot X_u = 0$:

$$\begin{aligned} 0 = \ddot\gamma \cdot X_u &= \frac{d}{dt}\left(\dot\gamma \cdot X_u\right) - \dot\gamma \cdot \dot X_u \text{ where } \dot\gamma(t) = X_u \dot u(t) + X_v \dot v(t) \\ &= \frac{d}{dt}\left(E\dot u + F\dot v\right) - \left(X_u \dot u + X_v \dot v\right) \cdot \left(X_{uu}\dot u + X_{uv}\dot v\right) \\ &= \frac{d}{dt}\left(E\dot u + F\dot v\right) - (X_u \cdot X_{uu})\dot u^2 - \left(X_u \cdot X_{uv} + X_v X_{uu}\right)\dot u \dot v - X_v \cdot X_{uv}\dot v^2, \end{aligned}$$

  where we used that $\dot\gamma \cdot X_u = \|X_u\|^2 \dot u + X_u \cdot X_v \dot v = E\dot u + F\dot v$.

- Now $X_u \cdot X_{uu} = \frac{1}{2}\frac{\partial}{\partial u}(X_u \cdot X_u) = \frac{1}{2}E_u$. Similarly, $X_u \cdot X_{uv} + X_v \cdot X_{uu} = \frac{\partial}{\partial u}(X_u \cdot X_v) = F_u$, and $X_v \cdot X_{uv} = \frac{1}{2}\frac{\partial}{\partial u}(X_v \cdot X_v) = \frac{1}{2}G_u$. Use these to simplify the above equation.
- Get $0 = \frac{d}{dt}\left(E\dot u + F\dot v\right) - \frac{1}{2}\left[E_u \dot u^2 + 2F_u \dot u \dot v + G_u \dot v^2\right]$.
- Likewise, our other equation, $\ddot\gamma \cdot X_v = 0$, yields $0 = \frac{d}{dt}\left(F\dot u + G\dot v\right) - \frac{1}{2}\left[E_v \dot u^2 + 2F_v \dot u \dot v + G_v \dot v^2\right]$.

The geodesic equations

The *geodesic equations* are any of the following three equivalent sets of equations along a curve $\gamma(t)$ on S:

- Vector form: $\nabla_{\gamma}\dot{\gamma} = \mathbf{0}$.
- Component form:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(E\dot{u} + F\dot{v}\right) - \frac{1}{2}\left[E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right] \\ 0 &= \frac{d}{dt}\left(F\dot{u} + G\dot{v}\right) - \frac{1}{2}\left[E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right] \end{aligned}$$

- Component form written using Christoffel symbols:

$$\begin{aligned} 0 &= \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 \\ 0 &= \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 \end{aligned}$$

- We've proved equivalence of the first two forms above. Equivalence of these with the third form is Proposition 7.4.5 of the text.

Existence and uniqueness of geodesics

Theorem

For each $p \in S$ and each $\mathbf{v} \in T_pS$ there is a unique maximal geodesic γ defined on an open interval $I \ni t_0$ such that $\gamma(t_0) = p$, $\dot{\gamma}(t_0) = \mathbf{v}$.

Proof:

- The equations

  $$\begin{aligned} 0 &= \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2 \\ 0 &= \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2 \end{aligned}$$

  have the form

  $$\begin{aligned} \ddot{u} &= f(u, v, \dot{u}, \dot{v}), \\ \ddot{v} &= g(u, v, \dot{u}, \dot{v}), \end{aligned}$$

  for smooth functions $f, g : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^4$.
- ODE theory: There is a unique solution of this system on an open interval I containing t_0, obeying initial conditions $u(t_0) = a$, $v(t_0) = b$, $\dot{u}(t_0) = c$, $\dot{v}(t_0) = d$.

Proof continued

- Write $\gamma(t) = X(u(t), v(t))$.
- Then $\dot{\gamma}(t) = X_u \dot{u} + X_v \dot{v}$.
- Initial data $\gamma(t_0) = p = X(u(t_0), u(t_0)) = X(a, b)$ give $u(t_0) = a$, $v(t_0) = b$.
- Initial data $\dot{\gamma}(t_0) = \mathbf{v} = cX_u + dX_v$ give $\dot{u}(t_0) = c$, $\dot{v}(t_0) = d$.
- Now all the conditions of the ODE existence and uniqueness theorem are satisfied. QED.

Example: Unit cylinder

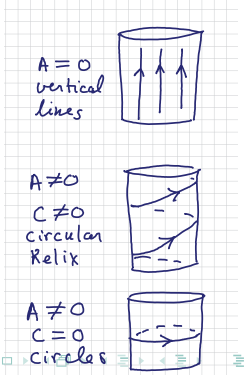
- Patch $X(u,v)=(\cos u, \sin u, v)$.
- $X_u=(-\sin u, \cos u, 0)$, $X_v=(0,0,1)$.
- Then $E=\|X_u\|^2=1$, $F=X_u\cdot X_v=0$, $G=\|X_v\|^2=1$
- Geodesic equations:

$$\begin{aligned}\frac{d}{dt}\left(E\dot{u}+F\dot{v}\right)-\frac{1}{2}\left[E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2\right]&=\ddot{u}=0,\\ \frac{d}{dt}\left(F\dot{u}+G\dot{v}\right)-\frac{1}{2}\left[E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2\right]&=\ddot{v}=0.\end{aligned}$$

- Solutions: $u(t)=At+B$, $v(t)=Ct+D$, for $A,B,C,D\in\mathbb{R}$.
- $\gamma(t)=X(u(t),v(t))=(\cos(At+B), \sin(At+B), Ct+D)$.

$\gamma(t) = (\cos(At + B), \sin(At + B), Ct + D)$

- For $A = 0$, get
  $\gamma(t) = (\cos B, \sin B, Ct + D)$.
  These are vertical lines.
- For $A \neq 0$ and $C \neq 0$, get
  $\gamma(t) = (\cos\tau, \sin\tau, k\tau + D')$ where
  $\tau := At + B$, $k = \frac{C}{A}$, $D' = D - \frac{BC}{A}$.
  This is a circular helix.
- $A \neq 0$ but $C = 0$, get
  $\gamma(t) = (\cos\tau, \sin\tau, D)$, where
  $\tau = At + B$
  These are circles.



Lecture 17: Minimizing the arclength

The geodesic equations: from last lecture

The *geodesic equations* are any of the following three equivalent sets of equations along a curve $\gamma(t)$ on S:

- Vector form: $\nabla_\gamma \dot{\gamma} = \mathbf{0}$.
- Component form:

$$0 = \frac{d}{dt}\left(E\dot{u} + F\dot{v}\right) - \frac{1}{2}\left[E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right]$$
$$0 = \frac{d}{dt}\left(F\dot{u} + G\dot{v}\right) - \frac{1}{2}\left[E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2\right]$$

- Component form written using Christoffel symbols:

$$0 = \ddot{u} + \Gamma^1_{11}\dot{u}^2 + 2\Gamma^1_{12}\dot{u}\dot{v} + \Gamma^1_{22}\dot{v}^2$$
$$0 = \ddot{v} + \Gamma^2_{11}\dot{u}^2 + 2\Gamma^2_{12}\dot{u}\dot{v} + \Gamma^2_{22}\dot{v}^2$$

First integral

- Define $g := E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$.
- Differentiate and use geodesic equations. Get $\dot{g} = 0$.
- Then $g = const$ along any geodesic.
- Therefore $\|\dot{\gamma}(t)\|$ is constant along any geodesic γ.
- For later convenience, multiply geodesic equations by $1/\sqrt{g}$, which can now be moved inside the t-derivative.

$$0 = \frac{d}{dt}\left(\frac{E\dot{u} + F\dot{v}}{\sqrt{g}}\right) - \frac{1}{2\sqrt{g}}\left[E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right]$$
$$0 = \frac{d}{dt}\left(\frac{F\dot{u} + G\dot{v}}{\sqrt{g}}\right) - \frac{1}{2\sqrt{g}}\left[E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right]$$

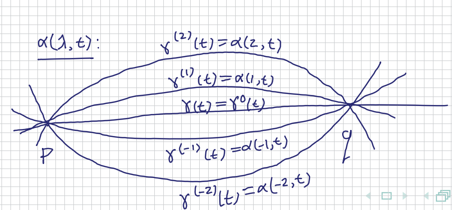
Minimizing curves

Definition

Say $\delta > 0$, $\epsilon > 0$. Consider a function $\alpha : (-\delta, \delta) \times (-\epsilon, \epsilon) \to \mathbb{R}^3$ with image contained in a single patch $X : U \to \mathbb{R}^3$ of surface S.

- For each $\lambda \in (-\delta, \delta)$ then $\gamma^{\lambda}(t) = \alpha(\lambda, t)$ is a curve.
- Say there is an a and a b with $-\epsilon < a < b < \epsilon$ and points $p, q \in S$ such that $\alpha(\lambda, a) = p$, $\alpha(\lambda, b) = q$ for all $\lambda \in (-\delta, \delta)$.
- Say that $\gamma^0(t) =: \gamma(t)$ is a geodesic from $p = \gamma(a)$ to $q = \gamma(b)$.

Then α is a *one parameter variation* of the geodesic $\gamma(t)$.



Arclength

- The arclength of the curve $\gamma^\lambda(t) = \alpha(\lambda, t)$, $t \in [a, b]$, is

$$L(\lambda) = \int_a^b \left\| \dot{\gamma}^\lambda(t) \right\| dt = \int_a^b \left\| \frac{\partial \alpha}{\partial t}(\lambda, t) \right\| dt.$$

- Minimize this function. The condition for the curve $\gamma(t) = \gamma^0(t)$ to be a critical point of L is

$$\begin{aligned} 0 = \frac{dL}{d\lambda}\Big|_{\lambda=0} &= \int_a^b \frac{\partial}{\partial \lambda}\Big|_{\lambda=0} \left(\left\| \frac{\partial \alpha}{\partial t}(\lambda, t) \right\| \right) dt \\ &= \int_a^b \frac{1}{2\sqrt{g}} \frac{\partial g}{\partial \lambda}\Big|_{\lambda=0} dt \end{aligned} \tag{1}$$

  where

$$g(\lambda, t) := \left\| \frac{\partial \alpha}{\partial t}(\lambda, t) \right\|^2 = \left\| \dot{\gamma}^\lambda(t) \right\|^2 = E\left(\dot{u}^\lambda\right)^2 + 2F\left(\dot{u}^\lambda\right)\left(\dot{v}^\lambda\right) + G\left(\dot{v}^\lambda\right)^2.$$

Differentiating g

- Last slide: We used the coordinate patch $X : U\mathbb{R}^3$ to describe the curve γ^λ using the curve $(u^\lambda(t), v^\lambda(t))$ in U, where $X(u^\lambda(t), v^\lambda(t)) = \gamma^\lambda(t)$.
- We wrote $g(\lambda, t) = \left\|\dot{\gamma}^\lambda(t)\right\|^2 = E\left(\dot{u}^\lambda\right)^2 + 2F\left(\dot{u}^\lambda\right)\left(\dot{v}^\lambda\right) + G\left(\dot{v}^\lambda\right)^2.$
- Now differentiate g (for simplicity, write $u = u^\lambda$, $v = v^\lambda$):

$$\begin{aligned}\frac{\partial g}{\partial \lambda} &= \frac{\partial}{\partial \lambda}\left[E\left(\dot{u}\right)^2 + 2F\left(\dot{u}\right)\left(\dot{v}\right) + G\left(\dot{v}\right)^2\right]\\ &= \left(E_u\frac{\partial u}{\partial \lambda} + E_v\frac{\partial v}{\partial \lambda}\right)\left(\dot{u}\right)^2 + 2E\dot{u}\frac{\partial^2 u}{\partial \lambda \partial t} + \dots\\ &= \left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right)\frac{\partial u}{\partial \lambda} + \left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right)\frac{\partial v}{\partial \lambda}\\ &\quad + 2\left(E\dot{u} + F\dot{v}\right)\frac{\partial^2 u}{\partial \lambda \partial t} + 2\left(F\dot{u} + G\dot{v}\right)\frac{\partial^2 v}{\partial \lambda \partial t}.\end{aligned} \tag{2}$$

- Plug this into equation (1) of previous slide.

...continued

- Plugging equation (2) into equation (1) produces

$$0 = \frac{dL}{d\lambda}\Big|_{\lambda=0} = \int_a^b \frac{1}{2\sqrt{g}} \left[\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2 \right) \frac{\partial u}{\partial \lambda} \right.$$

$$\left. + \left(E_v \dot{u}^2 + 2F_v \dot{u}\dot{v} + G_v \dot{v}^2 \right) \frac{\partial v}{\partial \lambda} \right] dt$$

$$+ \int_a^b \frac{1}{\sqrt{g}} \left[\left(E\dot{u} + F\dot{v} \right) \frac{\partial^2 u}{\partial \lambda \partial t} + \left(F\dot{u} + G\dot{v} \right) \frac{\partial^2 v}{\partial \lambda \partial t} \right] dt$$

- The integral in the final line can be integrated by parts. It becomes

$$- \int_a^b \left[\frac{\partial}{\partial t} \left(\frac{E\dot{u} + F\dot{v}}{\sqrt{g}} \right) \frac{\partial u}{\partial \lambda} + \frac{\partial}{\partial t} \left(\frac{F\dot{u} + G\dot{v}}{\sqrt{g}} \right) \frac{\partial v}{\partial \lambda} \right] dt$$

$$+ \frac{1}{\sqrt{g}} \left[\left(E\dot{u} + F\dot{v} \right) \frac{\partial u}{\partial \lambda} + \left(F\dot{u} + G\dot{v} \right) \frac{\partial v}{\partial \lambda} \right]_a^b$$

...continued

- We have $\frac{1}{\sqrt{g}}\left[\left(E\dot{u}+F\dot{v}\right)\frac{\partial u}{\partial\lambda}+\left(F\dot{u}+G\dot{v}\right)\frac{\partial v}{\partial\lambda}\right]_a^b=0$ because $\frac{\partial u}{\partial\lambda}(a)=\frac{\partial v}{\partial\lambda}(a)=\frac{\partial u}{\partial\lambda}(b)=\frac{\partial v}{\partial\lambda}(b)=0$.
- Putting everything else together, we can write $0=L'(0)=\left.\frac{dL}{d\lambda}\right|_{\lambda=0}=\int\limits_a^b\left[U\frac{\partial u}{\partial\lambda}+V\frac{\partial v}{\partial\lambda}\right]dt$ where

$$\begin{aligned}U&=\frac{1}{2\sqrt{g}}\left(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2\right)-\frac{\partial}{\partial t}\left(\frac{E\dot{u}+F\dot{v}}{\sqrt{g}}\right)\\ V&=\frac{1}{2\sqrt{g}}\left(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2\right)-\frac{\partial}{\partial t}\left(\frac{F\dot{u}+G\dot{v}}{\sqrt{g}}\right).\end{aligned}\tag{3}$$

- Key point: We require $0=L'(0)=\int\limits_a^b\left[U\frac{\partial u}{\partial\lambda}+V\frac{\partial v}{\partial\lambda}\right]dt$ *for all* $\frac{\partial u}{\partial\lambda}$ and $\frac{\partial v}{\partial\lambda}$. This can only happen if $U=0$ and $V=0$ (for a proof, see text).

The equations $U = 0$, $V = 0$

- From equations (3), the equations $U = 0$ and $V = 0$ are

$$\begin{aligned} 0 &= \frac{1}{2\sqrt{g}}\left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right) - \frac{\partial}{\partial t}\left(\frac{E\dot{u} + F\dot{v}}{\sqrt{g}}\right) \\ 0 &= \frac{1}{2\sqrt{g}}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right) - \frac{\partial}{\partial t}\left(\frac{F\dot{u} + G\dot{v}}{\sqrt{g}}\right). \end{aligned} \tag{4}$$

- But these are the *geodesic equations*!

Theorem

For all smooth curves γ from p to q in S, the arclength functions $L[\gamma]$ is a stationary point with respect to any one-parameter family of variations of γ on S if and only if γ is a geodesic of S.

Final remarks

- We worked on one patch $X : U \to \mathbb{R}^3$ with smooth curves. Simple to generalize to finitely many patches and to variations that can include piecewise smooth curves. The critical points are still (smooth) geodesics.
- Geodesics are critical points of arclength but not all geodesics are absolute of even local minima. Example:
    - Segments of great circles on spheres (intersections of the sphere with any plane through the origin) are geodesics.
    - No segment of a great circle that begins at the north pole and extends past the south pole can be a minimum, nor even a local minimum, of arclength.
- A geodesic is a *minimizing curve* or *minimizing geodesic* if it is a local minimum. Minimizing geodesics don't always exist in general, but will always exist if the surface is Cauchy complete.
- The idea of a geodesic can be extended beyond surfaces to Riemannian manifolds and to metric spaces.

Lecture 18: Gauss-Codazzi-Mainardi equations, Theorema Egregium

Gauss-Codazzi-Mainardi equations

Reminder:

- Surface S with surface patch $X : U \to \mathbb{R}^3$. $(u, v) \in U$.
- 1FF of patch is $E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$.
- 2FF of patch is $L(u, v)du^2 + 2M(u, v)dudv + N(u, v)dv^2$.
- Let $\mathbf{N}$ be the unit normal to the surface S.
- Gauss equations:

$$\begin{aligned} X_{uu} &= \Gamma^1_{11} X_u + \Gamma^2_{11} X_v + L\mathbf{N} \\ X_{uv} &= \Gamma^1_{12} X_u + \Gamma^2_{12} X_v + M\mathbf{N} \\ X_{vv} &= \Gamma^1_{22} X_u + \Gamma^2_{22} X_v + N\mathbf{N} \end{aligned}$$

- The Christoffel symbols depend only on the 1FF; e.g., $\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}$, etc.

A tedious calculation

- Differentiating the Gauss equations, compute $(X_{uu})_v$ and $(X_{uv})_u$.
- But partial derivatives commute, so $(X_{uu})_v = (X_{uv})_u$.
- In resulting equation, replace X_{uu}, X_{uv}, X_{vv} using Gauss equations again. Simplify. Get:

$$\begin{aligned} 0 = & \left(\frac{\partial \Gamma^1_{11}}{\partial v} - \frac{\partial \Gamma^1_{12}}{\partial u} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12} \right) X_u \\ & + \left(\frac{\partial \Gamma^2_{11}}{\partial v} - \frac{\partial \Gamma^2_{12}}{\partial u} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} \right) X_v \\ & + \left(L_v - M_u - \Gamma^1_{12} L + \Gamma^1_{11} M - \Gamma^2_{12} M + \Gamma^2_{11} N \right) \mathbf{N} \\ & + L\mathbf{N}_v - M\mathbf{N}_u \end{aligned} \tag{1}$$

- Use $\mathbf{N} \cdot X_u = \mathbf{N} \cdot X_v = 0$, $\mathbf{N} \cdot \mathbf{N}_u = \frac{1}{2}(\mathbf{N} \cdot \mathbf{N})_u = 0$, $\mathbf{N} \cdot \mathbf{N}_v = \frac{1}{2}(\mathbf{N} \cdot \mathbf{N})_v = 0$. Then

$$L_v - M_u - \Gamma^1_{12} L + \Gamma^1_{11} M - \Gamma^2_{12} M + \Gamma^2_{11} N = 0. \tag{2}$$

Codazzi-Mainardi equations

- Last slide:
$$L_v - M_u - \Gamma^1_{12} L + \Gamma^1_{11} M - \Gamma^2_{12} M + \Gamma^2_{11} N = 0.$$
- Can repeat the procedure, starting instead by computing $(X_{vu})_v$ and $(X_{vv})_u$ and subtracting them to get zero. Get
$$M_v - N_u - \Gamma^1_{22} L + \Gamma^1_{12} M - \Gamma^2_{22} M + \Gamma^2_{12} N = 0.$$
- The two equations above are named for Codazzi and Mainardi.
- But can also subtract the equation at the top (equation (2) from last slide) from equation (1) of the last slide. Get (1) with its third line removed:
$$\begin{aligned} 0 = &\left(\frac{\partial \Gamma^1_{11}}{\partial v} - \frac{\partial \Gamma^1_{12}}{\partial u} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12} \right) X_u \\ &+ \left(\frac{\partial \Gamma^2_{11}}{\partial v} - \frac{\partial \Gamma^2_{12}}{\partial u} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} \right) X_v \\ &+ L\mathbf{N}_v - M\mathbf{N}_u. \end{aligned} \tag{3}$$

Simplify $L\mathbf{N}_v - M\mathbf{N}_u$

- When we introduced Weingarten map (Lecture 11), we had $W(X_u) = -DG(X_u) = -\mathbf{N}_u$ where $G = \mathbf{N} =$ Gauss map.
- Similarly, $W(X_v) = -DG(X_u) = -\mathbf{N}_v$.
- Can express this using $E, \dots, N$ since in matrix notation $\mathcal{W} = \mathcal{F}_I^{-1}\mathcal{F}_{II}$.
- After some calculation, get $L\mathbf{N}_v - M\mathbf{N}_u = \frac{(NL-M^2)}{(EG-F^2)}\left[FX_u - EX_v\right] = K_G\left[FX_u - EX_v\right]$.
- Use this to substitute for $L\mathbf{N}_v - M\mathbf{N}_u$ in (3) (last slide).
- Resulting equation has X_u-component:

$$\frac{\partial \Gamma^1_{11}}{\partial v} - \frac{\partial \Gamma^1_{12}}{\partial u} + \Gamma^1_{22}\Gamma^2_{11} - \Gamma^1_{12}\Gamma^2_{12} + FK_G = 0.$$

- The X_v-component is

$$\frac{\partial \Gamma^2_{11}}{\partial v} - \frac{\partial \Gamma^2_{12}}{\partial u} + \Gamma^1_{11}\Gamma^2_{12} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - EK_G = 0.$$

Gauss curvature equations

- Last two equations were derived from $(X_{uu})_v - (X_{uv})_u = 0$. Get two more equations from $(X_{vu})_v - (X_{vv})_u = 0$.
- All four such equations are called the *Gauss equations* (we derived them by starting from another set of equations called Gauss equations).
- We can write the Gauss equations by isolating K_G:

$$\begin{aligned}
EK_G &= \left(\Gamma^2_{11}\right)_v - \left(\Gamma^2_{12}\right)_u + \Gamma^1_{11}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^2_{22} - \Gamma^1_{12}\Gamma^2_{11} - \Gamma^2_{12}\Gamma^2_{12},\\
FK_G &= \left(\Gamma^1_{12}\right)_u - \left(\Gamma^1_{11}\right)_v + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11},\\
FK_G &= \left(\Gamma^2_{12}\right)_v - \left(\Gamma^2_{22}\right)_u + \Gamma^1_{12}\Gamma^2_{12} - \Gamma^1_{22}\Gamma^2_{11},\\
GK_G &= \left(\Gamma^1_{22}\right)_u - \left(\Gamma^1_{12}\right)_v + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^1_{12}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}.
\end{aligned}$$

- And recall Codazzi-Mainardi:

$$\begin{aligned}
L_v - M_u &= \Gamma^1_{12}L - \Gamma^1_{11}M + \Gamma^2_{12}M - \Gamma^2_{11}N,\\
M_v - N_u &= \Gamma^1_{22}L - \Gamma^1_{12}M + \Gamma^2_{22}M - \Gamma^2_{12}N.
\end{aligned}$$

Ugly equations, beautiful results

- Four different equations for K_G. Therefore, there are identities amongst the right-hand sides, showing that they are all equal (these are called *Bianchi identities*).
- Combining the four Gauss equations, we get a determinant formula for K_G:

$$K_G = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}^2}$$

- $K_G = \det \mathcal{W} = \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I} = \kappa_1 \kappa_2$ *only depends on the first fundamental form of the surface!* This statement is often called the *Theorema Egregium* (remarkable theorem) of Gauss, though we will use the name for a corollary.

Relations between the 1FF and 2FF?

- $K_G = \det \mathcal{W} = \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I}$, so we now have a relation between the 1FF and 2FF.
- The Codazzi-Mainardi equations also relate the 1FF and 2FF.
- These are the only such relations. (If the 1FF completely determined the 2FF, one of the assumptions of the following theorem would be redundant.)

Theorem

If $X : U \to \mathbb{R}^3$ and $\tilde{X} : U \to \mathbb{R}^3$ are two surface patches with the same 1FF and 2FF, there is a direct isometry Φ of $\mathbb{R}^3$ such that $\tilde{X} = \Phi \circ X$.

This is an analogue for surfaces of the fundamental theorem for plane curves.

K_G for a surface of revolution

$$K_G = \frac{\begin{vmatrix} -\frac{1}{2}E_{vv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & -\frac{1}{2}E_v \\ -\frac{1}{2}G_u & E & 0 \\ \frac{1}{2}G_v & 0 & G \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & 0 \\ \frac{1}{2}G_u & 0 & G \end{vmatrix}}{\begin{vmatrix} E & 0 \\ 0 & G \end{vmatrix}^2}$$

$$= -\frac{1}{2\sqrt{EG}}\left[\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right) + \frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)\right] = \frac{1}{\sqrt{\det \mathcal{F}_I}}\,\mathrm{div}\left(-\frac{(G_u, E_v)}{2\sqrt{\det \mathcal{F}_I}}\right)$$

- Recall the divergence of a vector field $\mathbf{V} = (V^1, V^2)$ on U: $\mathrm{div}\,\mathbf{V} = \frac{\partial V^1}{\partial u} + \frac{\partial V^2}{\partial v}$.
- If $F = 0$ and also $E = 1$, get $K_G = -\frac{1}{2\sqrt{G}}\frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{G}}\right) = -\frac{1}{\sqrt{G}}\frac{\partial^2\sqrt{G}}{\partial u^2}$.
- Surface of revolution has 1FF $du^2 + f^2(u)dv^2$ (so $E = 1$, $G = f^2(u)$). Then $K_G = -\ddot{f}(u)/f(u)$.

Theorema Egregium

Theorem (Theorema Egregium)

The Gauss curvature is preserved by local isometries.

Proof.

- $f : S_1 \to S_2$ is a local isometry if it is a local diffeomorphism that maps any curve in S_1 to a curve of the same length in S_2.
- A local diffeo is a local isometry iff surface patches $X_1 : U \to \mathbb{R}^3$ for S_1 and $X_2 = f \circ X_1 : U \to \mathbb{R}^3$ for S_2 have same 1FF [text, Corollary 6.3.2].
- But K_G is completely determined by the 1FF.

□

- Meaning: Gives a necessary condition for two surfaces to have the same "local intrinsic geometry" (the 1FF); e.g., If two surfaces have different values of, say, $\sup K_G$, they cannot be isometric.
- Naive question: Is it a sufficient condition? In what sense?

Consequence for map making

Theorem

Any geographic map of the Earth's surface must distort distances.

Proof.

- Geographic maps are regions of planes. Planes have $\mathcal{W}_{II} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $K_G = \det \mathcal{W} = 0$.
- The Earth is (approximately) a round sphere, so $\mathcal{W}_{II} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $K_G = \det \mathcal{W} = 1$.
- $0 \neq 1$.

□

When we study the Gauss-Bonnet theorem, we will see that this argument does not require the Earth to be perfectly or approximately round.

Lecture 19: Minimal surfaces 1

Minimal surfaces

Definition

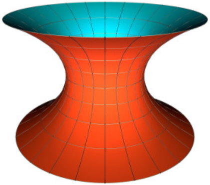
Consider a surface S.

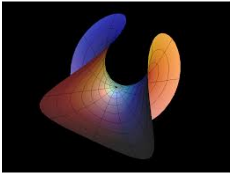
- If the 2FF of S vanishes everywhere, S is *totally geodesic*.
- If K_G vanishes everywhere (so $\det \mathcal{W} = 0$), S is *Gauss flat* or *intrinsically flat*.
- If the mean curvature vanishes everywhere $H = 0$, S is a *minimal surface*.
- A surface that minimizes area is a *least area surface*.

- Just as geodesics are extrema of the arclength, compact minimal surfaces are extrema of the area.
- Every least area surface is a minimal surface, but not every minimal surface is a least area surface.
- Minimal surfaces always minimize area compared to other surfaces which differ only in a "sufficiently small region".

A few examples

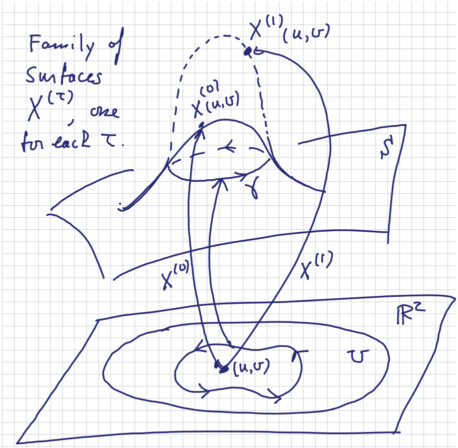
- Planes $ax + by + cz = d$.
- Catenoid
  $X(u, v) = (x(u, v), y(u, v), z(u, v))$
  $x = \cosh(u)\cos(v)$
  $y = \cosh(u)\sin(v)$
  $z = u$
- Enneper's surface
  $X(u, v) = (x(u, v), y(u, v), z(u, v))$
  $x(u, v) = \frac{u}{3}\left(1 - \frac{u^2}{3} + v^2\right)$
  $y(u, v) = \frac{v}{3}\left(1 - \frac{v^2}{3} + u^2\right)$
  $z(u, v) = \frac{1}{3}\left(u^2 - v^2\right)$





Variation of area

- Family of surface patches $X^{(\tau)} : U \to \mathbb{R}^3$ $\tau \in (-\delta, \delta)$ for $\delta > 0$.
- Require map $(u, v, \tau) \mapsto X^{(\tau)}(u, v)$ to be smooth.
- Define the *variation vector field* $\Phi := \frac{d}{d\tau}\Big|_{\tau=0} X^{(\tau)}(u, v) = \dot{X}^{(\tau)} big\big|_{\tau=0}$.
- γ is a simple closed curve containing interior region $\mathrm{int}(\gamma)$.
- Area of $\mathrm{int}(\gamma)$ is $A(\tau) = \int_{\mathrm{int}\,\gamma} dA_{X^{(\tau)}}$.



Variation of area: set-up

- $X^{(\tau)}(u,v) = X(\tau,u,v)$.
- The boundary curve doesn't vary: $\Phi(u,v) = 0$ if $X(\tau,u,v) = \gamma$.
- For each $X(\tau,u,v)$, we have the basis $\{X_u, X_v, \mathbf{N}\}$.
- Then $\Phi = a(\tau,u,v)\mathbf{N} + b(\tau,u,v)X_u + c(\tau,u,v)X_v$.
- Area $A(\tau) = \int_{\text{int}(\gamma)} \|X_u \times X_v\| \, dudv = \int_{\text{int}(\gamma)} \mathbf{N} \cdot (X_u \times X_v) \, dudv$.
- $\dot{A}(\tau) = \frac{dA}{d\tau} = \int_{\text{int}(\gamma)} \frac{\partial}{\partial\tau} \left(\mathbf{N} \cdot (X_u \times X_v)\right) dudv$.
- $\mathbf{N}$ is a unit vector, so $\dot{\mathbf{N}} \perp \mathbf{N}$. Therefore $\dot{\mathbf{N}} \perp X_u \times X_v$, so $\dot{\mathbf{N}} \cdot (X_u \times X_v) = 0$.
- Then $\dot{A}(\tau) = \frac{dA}{d\tau} = \int_{\text{int}(\gamma)} \mathbf{N} \cdot \frac{\partial}{\partial\tau} (X_u \times X_v) \, dudv$.
- Then $\dot{A}(\tau) = \frac{dA}{d\tau} = \int_{\text{int}(\gamma)} \mathbf{N} \cdot \left(\dot{X}_u \times X_v + X_u \times \dot{X}_v\right) dudv$, where $\dot{X} = \frac{\partial X}{\partial \tau}$.

First variation of area

- Last slide: $\dot{A}(\tau) = \frac{dA}{d\tau} = \int_{\mathrm{int}(\gamma)} \mathbf{N} \cdot \left(\dot{X}_u \times X_v + X_u \times \dot{X}_v \right) dudv$, where $\dot{X} = \frac{\partial X}{\partial \tau}$.
- Calculation in text p 310 then gives $\left.\frac{dA}{d\tau}\right|_{\tau=0} = \int_{\mathrm{int}(\gamma)} \left[\left(b\sqrt{EG-F^2} \right)_u + \left(c\sqrt{EG-F^2} \right)_v - 2a\left(EG-F^2\right) H \right] dudv$, where $H = \frac{LG-2MF+NE}{2(EG-F^2)}$.
- Use Green's theorem $\int_{\mathrm{int}(\gamma)} \left(\frac{\partial g}{\partial u} - \frac{\partial f}{\partial v} \right) dudv = \int_\gamma (fdu + gdv)$.
- Get $\left.\frac{dA}{d\tau}\right|_{\tau=0} = \int_\gamma \sqrt{EG-F^2}(bdv - cdu) - 2\int_{\mathrm{int}(\gamma)} aH\left(EG-F^2\right) dudv$
- $\Phi = 0$ along γ, so $b = c = 0$ in line integral along γ.
- *First variation of area formula*: $\left.\frac{dA}{d\tau}\right|_{\tau=0} = -2\int_{\mathrm{int}(\gamma)} aH\left(EG-F^2\right) dudv = -2\int_{\mathrm{int}(\gamma)} aH\sqrt{\det \mathcal{F}_I} dudv$.

Lecture 20: Plateau's problem, minimal surfaces 2

Plateau's problem

- First variation of area: $\frac{dA}{d\tau}\Big|_{\tau=0} = -2\int_{\mathrm{int}(\gamma)} aH\sqrt{\det \mathcal{F}_I}\,dudv$.
- $a = a(0, u, v)$ is the normal component of the variation vector field $\Phi = a\mathbf{N} + bX_u + cX_v$ (last lecture).
- Stationary points: $\frac{dA}{d\tau}\Big|_{\tau=0} = 0$ for all a iff $H \equiv 0$ on $\mathrm{int}(\gamma)$.
- Plateau's problem: Given a simple closed curve $\gamma : [\alpha, \beta] \to \mathbb{R}^3$, find a least area surface whose boundary is γ.
- Step 1: Find the minimal surfaces (the critical points of area) spanning γ. These are minimal surfaces $H = 0$.
- Soap films spanning a ring are solutions of Plateau's problem
- Soap bubbles are not usually solutions of Plateau's problem. Bubbles are supported by air pressure, and are *CMC surfaces* (constant mean curvature surfaces). They obey $H = c = const$, so minimal surfaces $H = 0$ are a special case.

Nonpositive curvature

Theorem (Gaussian curvature of minimal surfaces)

Minimal surfaces in $\mathbb{R}^3$ *have* $K_G \leq 0$.

Proof.

- If S is a minimal surface then $H = 0$ at each point of S.
- $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ ($\kappa_i =$ principal curvatures).
- Then κ_1 and κ_2 have opposite signs at each point, or one of them is zero; so their product is negative, or zero.
- Then $K_G = \kappa_1 \kappa_2$ must be negative, or zero.

□

No compact minimal surfaces in $\mathbb{R}^3$

- Recall a surface $S \subset \mathbb{R}^3$ is compact if it is bounded so it lies within some sphere, and complete so Cauchy sequences converge. (Some definitions also require no boundary, but that follows from our definition of a surface.)
- A sphere is compact. So is a torus. A punctured sphere (i.e., a sphere minus a point) is not compact. A plane is not compact.

Theorem

There are no compact minimal surfaces embedded in $\mathbb{R}^3$.

Proof.

- At every point of a minimal surface, $0 = 2H = \kappa_1 + \kappa_2$, so the principal curvatures have opposite signs or are both zero.
- Then $K_G = \kappa_1 \kappa_2 \leq 0$ at every point.
- But every compact surface has at least one point where $K_G > 0$.

☐

Example of a minimal surface: A catenoid

Consider the catenoid $\cosh z = \sqrt{x^2 + y^2}$ parametrized by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u) .$$

Exercise:

- Compute that $\mathcal{F}_I = \cosh^2 u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Compute that $\mathcal{F}_{II} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Conclude that $\mathcal{W} = \mathcal{F}_I^{-1} \mathcal{F}_{II} = \operatorname{sech}^2 u \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Then $\kappa_1 = -\operatorname{sech}^2 u$, $\kappa_2 = \operatorname{sech}^2 u$.
- $H = \frac{1}{2}(\kappa_1 + \kappa_2) = 0$, so this is a minimal surface.
- $K_G = \kappa_1 \kappa_2 = -\operatorname{sech}^4 u < 0$.

Minimal surfaces of revolution

Theorem

Any minimal surface that is a surface of revolution is an open subset of a catenoid or a plane.

- To begin proof, recall surface of revolution. If necessary, use an isometry so that surface is revolved around z-axis.
- Then $X = (f(u)\cos v, f(u)\sin v, g(u))$.
- Choose profile curve to be unit speed: $\dot{f}^2(u) + \dot{g}^2(u) = 1$.
- We had $\mathcal{F}_I = \begin{bmatrix} 1 & 0 \\ 0 & f^2 \end{bmatrix}$, $\mathcal{F}_{II} = \begin{bmatrix} \dot{f}\ddot{g} - \ddot{f}\dot{g} & 0 \\ 0 & f\dot{g} \end{bmatrix}$, $\mathcal{W} = \begin{bmatrix} \dot{f}\ddot{g} - \ddot{f}\dot{g} & 0 \\ 0 & \dot{g}/f \end{bmatrix}$.
- This is a minimal surface iff $0 = H = \dot{f}\ddot{g} - \ddot{f}\dot{g} + \dot{g}/f$.

Proof continued

- Must solve ODE system $\dot{f}^2 + \dot{g}^2 = 1$, $\dot{f}\ddot{g} - \ddot{f}\dot{g} + \dot{g}/f = 0$.
- Possibilities:
    1. $\dot{g} = 0$ on open interval.
    2. $\dot{f} = 0$ on open interval.
    3. $\dot{f} \neq 0$, $\dot{g} \neq 0$ except perhaps at isolated points.
- Possibility 1: Then $g = k = const$ and $\dot{f}^2 = 1$ so $f = \pm u + u_0$, $u_0 = const$. The 1FF is $du^2 + (\pm u + u_0)^2 dv^2 = 0$. Writing $r := \pm u + u_0$ and $\theta := v$, get $dr^2 + r^2 d\theta^2$. This is the 1FF of a horizontal plane in polar coordinates.
- Possibility 2: $\dot{f} = 0$ on open interval, then $f(u) = k = const$ and $\dot{g}^2 = 1$. But then $H = \dot{f}\ddot{g} - \ddot{f}\dot{g} + \dot{g}/f = 0 + 0 + 1/k \neq 0$. No solution.
- Possibility 3: Differentiate $\dot{f}^2 + \dot{g}^2 = 1$ to get $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$. Use this in H on next page.

Proof continued

- We have $\dot{f}\ddot{f} + \dot{g}\ddot{g} = 0$ and $0 = \dot{g}H = \dot{f}\dot{g}\ddot{g} - \ddot{f}\dot{g}^2 + \dot{g}^2/f$.
- Combining these, then

$$\begin{aligned} 0 &= -\dot{f}^2\ddot{f} - \ddot{f}\dot{g}^2 + \dot{g}^2/f = -\ddot{f}\left(\dot{f}^2 + \dot{g}^2\right) + \dot{g}^2/f = -\ddot{f} + \dot{g}^2/f \\ &= -\ddot{f} + \frac{1}{f}\left(1 - \dot{f}^2\right). \end{aligned}$$

- Multiplying by $-f$, we get $0 = f\ddot{f} + \dot{f}^2 - 1 = \frac{1}{2}\frac{d^2}{du^2}\left(f^2\right) - 1$.
- Then $\frac{d^2}{du^2}\left(f^2\right) = 2$, so $f^2(u) = u^2 + au + b$.
- A translation of u removes the au term. We choose $b = c^2 > 0$ so that $f^2(0) > 0$.
- Then $f^2(u) = u^2 + c^2$, so $\dot{f} = \frac{u}{\sqrt{u^2+c^2}}$.
- Then $\dot{g} = \frac{c}{\sqrt{u^2+c^2}}$ and $g(u) = c \operatorname{arcsinh} \frac{u}{c}$.

End of proof

We need only do a little rewriting:

- Define $\tilde{u} := g(u) = c \operatorname{arcsinh} \frac{u}{c}$, so $u = c \sinh \frac{\tilde{u}}{c}$.
- Then $f^2(u) = u^2 + c^2 = c^2 \left(\sinh^2 \frac{\tilde{u}}{c} + 1\right) = c^2 \cosh^2 \frac{\tilde{u}}{c}$, so $f(u) = c \cosh \frac{\tilde{u}}{c}$.
- Then

$$\begin{aligned} X(u,v) &= (f(u)\cos v, f(u)\sin v, g(u)) \\ &= \left(c \cosh \frac{\tilde{u}}{c} \cos v, c \cosh \frac{\tilde{u}}{c} \sin v, \tilde{u}\right). \end{aligned}$$

- This is a catenoid.

The minimal graph equation

- Graph $z = f(x, y)$.
- Mean curvature is $H = \frac{\left(1+f_y^2\right)f_{xx} - 2f_x f_y f_{xy} + \left(1+f_x^2\right)f_{yy}}{2\left(1+f_x^2+f_y^2\right)^{3/2}}$ (text, exercise 8.1.1).
- The equation $\left(1+f_y^2\right)f_{xx} - 2f_x f_y f_{xy} + \left(1+f_x^2\right)f_{yy} = 0$ is called the *minimal graph equation* (or *minimal surface equation*).

Theorem

Consider solutions of the minimal graph equation of the form $f(x, y) = F(x) + G(y)$. Up to isometry, the only solutions are planes and Scherk's surface $z = \ln \frac{\cos y}{\cos x}$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

Lecture 21: Local Gauss-Bonnet (one patch)

Gauss Bonnet theorem 1: single patch

Theorem (Gauss Bonnet for a single surface patch)

- *Let* $X : U \to \mathbb{R}^3$ *be a surface patch covering surface* S*.*
- *Let* $\gamma(s)$ *be a simple closed curve separating* S *into two regions, the interior* $\text{int}(\gamma)$ *and the exterior* $\text{ext}(\gamma)$*.*
- *Let* s *be a unit speed parameter for* γ*.*
- *Let* κ_g *be the geodesic curvature of* γ*.*
- *Let* K_G *be the Gauss curvature of* S*.*

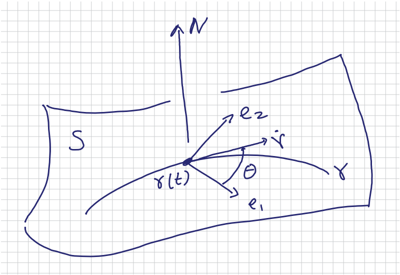
Then

$$\int_{\text{int}(\gamma)} K_G \, dA_X + \oint_\gamma \kappa_g ds = 2\pi.$$

- Compare: Hopf's umlaufsatz: $\oint_\gamma \kappa ds = 2\pi$ for a simple closed curve in $\mathbb{R}^2$.
- Follows from Green's theorem $\int_{\text{int}(\gamma)} (Q_u - P_v) du dv = \oint_\gamma P du + Q dv$.

Orthonormal basis: ONB

- Surface S with normal $\mathbf{N}$, patch $X : U \to \mathbb{R}^3$.
- $\{X_u, X_v, \mathbf{N}\}$ is not an ONB.
- Curve γ in S has tangent $\dot{\gamma}$.
- Choose $\mathbf{e}_1, \mathbf{e}_2 \in T_{\gamma(t)}S$, $\mathbf{e}_1 \perp \mathbf{e}_2$.
- Then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{N}\}$ is an ONB along γ.



$\dot{\gamma}$ and $\ddot{\gamma}$

- Take $\gamma(s)$ to be unit speed.
- $\dot{\gamma}$ makes angle θ with $\mathbf{e}_1$.
- $\dot{\gamma} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$.
- $\implies \mathbf{N}\times\dot{\gamma} = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$.
- $\ddot{\gamma} = \cos\theta\dot{\mathbf{e}}_1 + \sin\theta\dot{\mathbf{e}}_2 + \dot{\theta}\left(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2\right)$.
- Remark: We have not assumed a "transport law" for the $\mathbf{e}_i$ along γ, except that they remain tangent to S, orthonormal to each other, and differentiable wrt the parameter s along γ.
- $\kappa_g = \ddot{\gamma}\cdot(\mathbf{N}\times\dot{\gamma})$ if γ is unit speed.
- Can use this to compute that

$$\begin{aligned}\kappa_g &= \dot{\theta}\left(\sin^2\theta + \cos^2\theta\right) + \left(\cos\theta\dot{\mathbf{e}}_1 + \sin\theta\dot{\mathbf{e}}_2\right)\cdot\left(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2\right)\\ &= \dot{\theta} + \sin\theta\cos\theta\left(\dot{\mathbf{e}}_2\cdot\mathbf{e}_2 - \dot{\mathbf{e}}_1\cdot\mathbf{e}_1\right) + \cos^2\dot{\mathbf{e}}_1\cdot\mathbf{e}_2 - \sin^2\theta\dot{\mathbf{e}}_2\cdot\mathbf{e}_1.\end{aligned}$$

Simplify

- $\kappa_g = \dot{\theta} + \sin\theta\cos\theta\left(\dot{\mathbf{e}}_2 \cdot \mathbf{e}_2 - \dot{\mathbf{e}}_1 \cdot \mathbf{e}_1\right) + \cos^2 \dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 - \sin^2\theta \dot{\mathbf{e}}_2 \cdot \mathbf{e}_1$.
- Two easy simplifications:
    - $\dot{\mathbf{e}}_i \cdot \mathbf{e}_i = \frac{1}{2}\frac{d}{ds}(\mathbf{e}_i \cdot \mathbf{e}_i) = \frac{1}{2}\frac{d}{ds}(1) = 0$, and
    - $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 = \frac{d}{ds}\left(\mathbf{e}_1 \cdot \mathbf{e}_2\right) - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 = -\mathbf{e}_1 \cdot \dot{\mathbf{e}}_2$.
- So we get $\kappa_g = \dot{\theta} - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2$.
- Integrate the result around closed curve γ:

$$\begin{aligned}\oint_\gamma \kappa_g ds &= \oint_\gamma \dot{\theta} ds - \oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds\\ &= 2\pi - \oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds,\end{aligned}$$

  using $\oint_\gamma \dot{\theta} ds = \theta\Big|_0^{2\pi} = 2\pi$.

- Next: Convert last term on right to area integral of K_G (use Green's theorem).

Dealing with $\oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds$

- Chain rule: $\dot{\mathbf{e}}_2 = (\partial_u \mathbf{e}_2)\dot{u} + (\partial_v \mathbf{e}_2)\dot{v}$.
- Then $\oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds = \oint_\gamma \left[\mathbf{e}_1 \cdot (\partial_u \mathbf{e}_2)\dot{u} + \mathbf{e}_1 \cdot (\partial_v \mathbf{e}_2)\dot{v}\right] ds$.
- Line integral form: $\oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds = \oint_\gamma (\mathbf{e}_1 \cdot \partial_u \mathbf{e}_2) du + (\mathbf{e}_1 \cdot \partial_v \mathbf{e}_2) dv$.
- Use Green's theorem: $\oint_\gamma P du + Q dv = \int_{\text{int}(\gamma)} (Q_u - P_v) du dv$.
- Get $\oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds = \int_{\text{int}(\gamma)} \left[\partial_u \left(\mathbf{e}_1 \cdot \partial_v \mathbf{e}_2\right) - \partial_v \left(\mathbf{e}_1 \cdot \partial_u \mathbf{e}_2\right)\right] du dv$.
- Expand/simplify: $\oint_\gamma \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 ds = \int_{\text{int}(\gamma)} \left[(\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2) - (\partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2)\right] du dv$.
- So now we have

$$\oint_\gamma \kappa_g ds = 2\pi - \int_{\text{int}(\gamma)} \left[(\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2) - (\partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2)\right] du dv.$$

The final lemma

Last slide: $\oint_\gamma \kappa_g ds = 2\pi - \int_{\mathrm{int}(\gamma)} [(\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2) - (\partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2)]\, dudv$.

Lemma

- *Let the 1FF of the patch* $X : U \to \mathbb{R}^3$ *be* $Edu^2 + 2Fdudv + Gdv^2$.
- *Let the 2FF of the patch* $X : U \to \mathbb{R}^3$ *be* $Ldu^2 + 2Mdudv + Ndv^2$.

Then $\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2 \ - \partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2 = \frac{LN-M^2}{\sqrt{EG-F^2}} = K_G \sqrt{EG - F^2} = K_G \sqrt{\det \mathcal{F}_I}$.

We must prove this, but first:

Corollary (Gauss-Bonnet for one patch X)

$\oint_\gamma \kappa_g ds = 2\pi - \int_{\mathrm{int}(\gamma)} K_G \sqrt{\det \mathcal{F}_I} dudv = 2\pi - \int_{\mathrm{int}(\gamma)} K_G dA_X$.

Proof.

Plug the result of the theorem into the equation at the top of the slide. ☐

Proof of the lemma

- Idea: Write $\partial_u \mathbf{e}_i$, $\partial_v \mathbf{e}_i$ in the $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{N}\}$ orthonormal basis.
- Simplification: $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$, so $\partial_u(\mathbf{e}_1 \cdot \mathbf{e}_1) = 2\mathbf{e}_1 \cdot \partial_u \mathbf{e}_1 = 0$.
- Likewise $\mathbf{e}_1 \cdot \partial_v \mathbf{e}_1 = 0$, $\mathbf{e}_2 \cdot \partial_u \mathbf{e}_2 = 0$, $\mathbf{e}_2 \cdot \partial_v \mathbf{e}_2 = 0$.
- Then $\partial_u \mathbf{e_i}$ and $\partial_v \mathbf{e_i}$ have no $\mathbf{e}_i$ component, so:
  - $\partial_u \mathbf{e}_1 = a\mathbf{e}_2 + c\mathbf{N}$,
  - $\partial_v \mathbf{e}_1 = b\mathbf{e}_2 + d\mathbf{N}$,
  - $\partial_u \mathbf{e}_2 = -f\mathbf{e}_1 + g\mathbf{N}$,
  - $\partial_v \mathbf{e}_2 = -h\mathbf{e}_1 + k\mathbf{N}$,
- for coefficients $a, \dots, k$ (the minus signs are for later convenience).
- Then we get

$$\begin{aligned} &\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2 - \partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2 \\ &= (a\mathbf{e}_2 + c\mathbf{N}) \cdot (-h\mathbf{e}_1 + k\mathbf{N}) - (-f\mathbf{e}_1 + g\mathbf{N}) \cdot (b\mathbf{e}_2 + d\mathbf{N}) \qquad (1) \\ &= ck - dg. \end{aligned}$$

Proof of lemma continued...

- $\mathbf{N} = \mathbf{e_1} \times \mathbf{e}_2$ (since $\{\mathbf{e_1}, \mathbf{e}_2, \mathbf{N}\}$ is right-handed ONB).
- Then $(\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = (\mathbf{N}_u \times \mathbf{N}_v) \cdot (\mathbf{e_1} \times \mathbf{e}_2)$.
- $\implies (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = (\mathbf{N}_u \cdot \mathbf{e}_1)(\mathbf{N}_v \cdot \mathbf{e}_2) - (\mathbf{N}_u \cdot \mathbf{e}_2)(\mathbf{N}_v \cdot \mathbf{e}_1)$ (identity).
- $\implies (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = (\mathbf{N} \cdot \partial_u \mathbf{e}_1)(\mathbf{N} \cdot \partial_v \mathbf{e}_2) - (\mathbf{N} \cdot \partial_u \mathbf{e}_2)(\mathbf{N} \cdot \partial_v \mathbf{e}_1)$ (Leibniz).
- Use from last slide that:
  - $\partial_u \mathbf{e}_1 = a\mathbf{e}_2 + c\mathbf{N}$,
  - $\partial_v \mathbf{e}_1 = b\mathbf{e}_2 + d\mathbf{N}$,
  - $\partial_u \mathbf{e}_2 = -f\mathbf{e}_1 + g\mathbf{N}$,
  - $\partial_v \mathbf{e}_2 = -h\mathbf{e}_1 + k\mathbf{N}$,
- $\implies (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = ck - dg$.
- Inserting this into (1) from the last slide, we have

$$\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2 - \partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2 = (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N}. \tag{2}$$

End of proof

- Last slide: $\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2 - \partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2 = (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N}$.
- Chapter 8 (Lecture 15): $\mathbf{N}_u \times \mathbf{N}_v = K_G X_u \times X_v$.
- Then $(\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = K_G (X_u \times X_v) \cdot \mathbf{N} = K_G (X_u \times X_v) \cdot \frac{(X_u \times X_v)}{\|(X_u \times X_v)\|}$.
- $\implies (\mathbf{N}_u \times \mathbf{N}_v) \cdot \mathbf{N} = K_G \sqrt{\det \mathcal{F}_I} = K_G \|X_u \times X_v\| = K_G \sqrt{EG - F^2}$.
- Then we conclude that

$$\partial_u \mathbf{e}_1 \cdot \partial_v \mathbf{e}_2 - \partial_v \mathbf{e}_1 \cdot \partial_u \mathbf{e}_2 = K_G \sqrt{EG - F^2} = K_G \sqrt{\det \mathcal{F}_I},$$

which proves the lemma, and Gauss-Bonnet follows as a corollary.

A consequence of $\int_{\mathrm{int}(\gamma)} K_G dA_X + \oint_\gamma \kappa_g ds = 2\pi$.

Lemma

Let S be a surface covered by a single patch and bounded by a closed geodesic γ. Then S cannot have everywhere nonpositive Gauss curvature.

Proof.

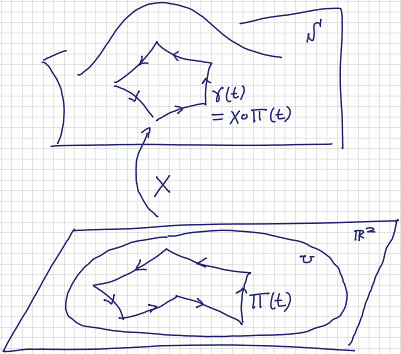
Geodesics have $\kappa_g = 0$, so $\int_{\mathrm{int}(\gamma)} K_G dA_X = 2\pi$, so $K_G > 0$ somewhere on S. ☐

Recall we already had a similar result for compact surfaces without boundary.

Lecture 22: Gauss-Bonnet for curvilinear polygons

Curvilinear polygons

- A *curvilinear polygon* is a region in $\mathbb{R}^2$ bounded by edges that meet at corners.
- We denote the boundary curve by $\Pi(t)$.
- One surface patch $X : U \to \mathbb{R}^3$, for simplicity only.
- Use X to lift it up to a region in surface S, bounded by curve $\gamma = X \circ \Pi$.



Curvilinear polygon definition

Definition

A *curvilinear polygon* is a continuous map $\Pi : \mathbb{R} \to \mathbb{R}^2$ such that, for some $T \in \mathbb{R}$ and a partition $0 = t_0 < t_1 < \cdots < t_n = T$, we have the following:

(i) (Boundary curve is closed:) $\Pi(t) = \Pi(t')$ if and only if $t - t'$ is an integer multiple of T.

(ii) (Boundary curve is smooth between finitely many corners:) Π is smooth at any $t \in (t_{i-1}, t_i)$, $i = 1, \ldots, n$.

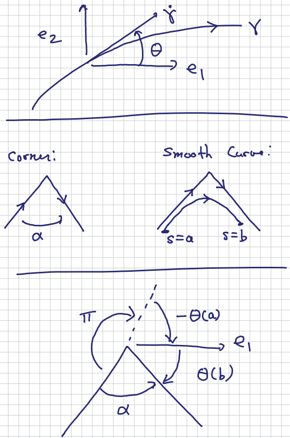
(iii) (Corners form well-defined angles:) The one-sided derivatives

$$\dot{\Pi}^{+}(t_i) = \lim_{t \nearrow t_i} \frac{\Pi(t) - \Pi(t_i)}{t - t_i}\ ,\ \dot{\Pi}^{-}(t_{i-1}) = \lim_{t \searrow t_{i-1}} \frac{\Pi(t) - \Pi(t_{i-1})}{t - t_{i-1}}\ ,$$

exist for each $i = 1, \ldots, n$.

Rounding off corners

- $\gamma(t)$ a unit speed curve.
- $\{\mathbf{e}_1, \mathbf{e}_2\}$ an ONB.
- $\dot{\gamma} = \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2$.
- Plane curves: $\kappa_g = \kappa_S = \dot{\theta}$.
- Curve in diagram smooths out corner with angle α.
- Along the smooth curve: $\int_a^b \dot{\theta} ds = \theta(b) - \theta(a) = \pi - \alpha$.
- We will consider when $\alpha \in [0, 2\pi)$ is an interior angle in polygon.



$$\alpha = 2\pi - [\pi - \theta(a) + \theta(b)]$$
$$= \pi - (\theta(b) - \theta(a))$$

Gauss-Bonnet for curvilinear polygons

$$\int_{\mathrm{int}(\gamma)} K_G dA_X + \sum_{i=1}^{n} \int_{\gamma_i} \kappa_g ds + \sum_{i=1}^{n} (\pi - \alpha_i) = 2\pi \ ,$$

where

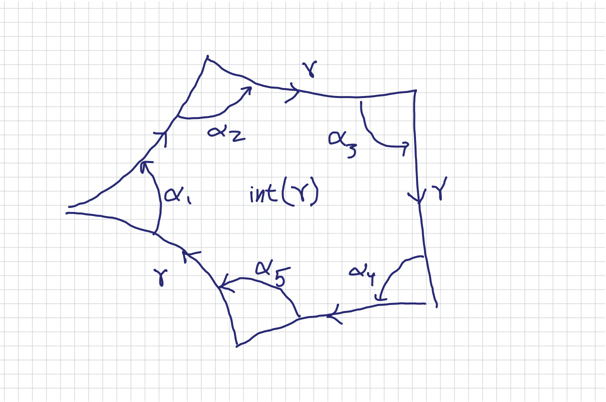
- γ is a simple, closed, unit speed curve bounding a curvilinear polygon in a patch X of surface S,
- γ is a union of n smooth segments γ_i which meet at n vertices $\gamma(t_i)$, $i = 1, \ldots, n$, and
- $\alpha_i \in [0, 2\pi)$ is the interior angle at the i^{th} vertex.

We can also write this formula as

$$\int_{\mathrm{int}(\gamma)} K_G dA_X + \int_{\gamma} \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi \ ,$$

where $\int_\gamma$ means $\sum\limits_{i=1}^{n} \int_{\gamma_i}$.

Gauss-Bonnet illustrated



$$\int_{\mathrm{int}(\gamma)} K_G dA_X + \sum_{i=1}^{n} \int_{\gamma_i} \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi \ .$$

$$\int_{\text{int}(\gamma)} K_G dA_X + \sum_{i=1}^{n} \int_{\gamma_i} \kappa_g ds = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$

Special case:

- If γ consists of geodesic segments γ_i then

$$\int_{\text{int}(\gamma)} K_G dA_X = \sum_{i=1}^{n} \alpha_i - (n-2)\pi \ .$$

Corollary

- *The total curvature of a hemisphere of any radius is* $\int K_G dA = 2\pi$*.*
- *The total curvature of a sphere of any radius is* $\int K_G dA = 4\pi$*.*

Proof.

- Hemisphere: Boundary curve γ is the equator, which is a geodesic.
- Sphere: Add two hemispheres.

☐

Triangulations of a surface S

Definition

A *triangulation* of S is a collection of curvilinear polygons such that

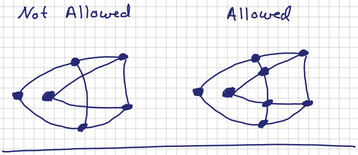
(i) every point of S is in at least one polygon,

(ii) any two polygons are either disjoint or intersect at a common vertex or along a common edge, and

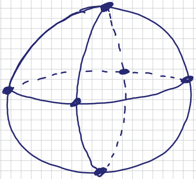
(iii) each edge is an edge of exactly two polygons.

Theorem

Every compact surface can be triangulated by finitely many curvilinear polygons.



Sphere triangulated
by its octants



8 faces (octants)
12 edges
6 vertices

Euler number (Euler characteristic)

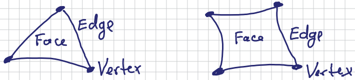
- Curvilinear polygons have (i) faces, (ii) edges, and (iii) vertices.
- A triangulation will have (ii) F faces, (ii) E edges, and (iii) V vertices.

Definition

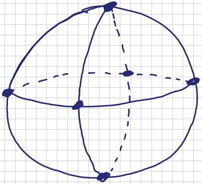
The *Euler number* (or *Euler characteristic*) of a triangulation of a surface S is

$$\chi := F - E + V.$$

e.g., Sphere triangulated by its octants has $\chi = 8 - 12 + 6 = 2$.



Sphere triangulated
by its octants



8 faces (octants)
12 edges
6 vertices

$F - E + V = 8 - 12 + 6 = 2$

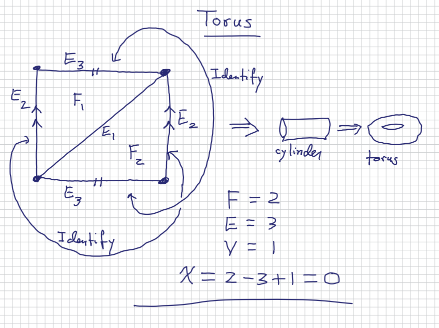
Gauss-Bonnet theorem for compact surfaces

Theorem

For any compact surface S we have

$$2\pi\chi = \int_S K_G dA.$$

- Corollary: χ depends only on the surface S, not on the triangulation.
- We write $\chi = \chi(S)$,
- For any sphere, we have $\chi(\mathbb{S}^2) = 2$.
- For any torus, we have $\chi(\mathbb{T}^2) = 0$.



$F = 2$
$E = 3$
$V = 1$

$\chi = 2 - 3 + 1 = 0$

Lecture 23 Gauss-Bonnet for compact surfaces

Proof of Gauss-Bonnet theorem for compact surfaces

Theorem

For any compact surface S we have $2\pi\chi = \int_S K_G dA$.

Proof:

- Consider a triangulation with
  - faces f_i, $i = 1, \ldots, F$,
  - edges e_j, $j = 1, \ldots, E$, and
  - vertices v_k, $k = 1, \ldots, V$.
- Choose the faces f_i small enough so each "triangle" (i.e., curvilinear polygon) fits in one patch $X_i : U_i \to \mathbb{R}$.
- Say face f_i is a polygon with p_i edges e_{im} and p_i vertices v_{in}, $m, n = 1, \ldots, p_i$. Let α_{in} be the interior angle of vertex v_{in}. Then

$$\int_S K_G dA = \sum_{i=1}^{F} \int_{f_i} K_G dA_{X_i} = \sum_{i=1}^{F} \left\{ - \sum_{m=1}^{p_i} \int_{e_{im}} \kappa_g ds - \sum_{n=1}^{p_i} (\pi - \alpha_{in}) + 2\pi \right\} .$$

Proof continued

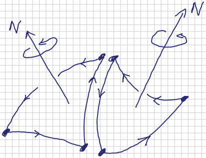
$$\int_S K_G dA = -\sum_{i=1}^{F}\sum_{m=1}^{p_i}\int_{e_{im}}\kappa_g ds - \sum_{i=1}^{F}\sum_{n=1}^{p_i}(\pi - \alpha_{in}) + \sum_{i=1}^{F} 2\pi.$$

- Last term: $\sum_{i=1}^{F} 2\pi = 2\pi F$.
- Middle term:
  - $\sum_{i=1}^{F}\sum_{n=1}^{p_i}\alpha_{in}$ is the sum over every face of every interior angle in that face.
  - Same as sum over every vertex of every interior angle at that vertex, which is therefore $2\pi V$.
  - Also, $-\sum_{i=1}^{F}\sum_{n=1}^{p_i}\pi = -\pi\sum_{i=1}^{F}\sum_{n=1}^{p_i} 1$, and $\sum_{i=1}^{F}\sum_{n=1}^{p_i} 1$ equals twice the number of edges, since each edge belongs to two faces and so is counted twice.
  - Then $-\sum_{i=1}^{F}\sum_{n=1}^{p_i}(\pi - \alpha_{in}) = -2\pi E + 2\pi V$.
  - Collect results: $\int_S K_G dA = -\sum_{i=1}^{F}\sum_{m=1}^{p_i}\int_{e_{im}}\kappa_g ds + 2\pi(-E + V + F)$.

Proof continued

$$\int_S K_G dA = -\sum_{i=1}^{F}\sum_{m=1}^{p_i}\int_{e_{im}}\kappa_g ds + 2\pi(-E+V+F).$$

- But $\sum_{i=1}^{F}\sum_{m=1}^{p_i}\int_{e_{im}}\kappa_g ds = 0$ because, when summing over faces, each edge e_{im} is counted twice, once for each face to which is belongs, but with opposite orientations.
- $\kappa_g = \ddot{\gamma}\cdot(\mathbf{N}\times\dot{\gamma})$, so $s \mapsto u = -s \implies \kappa_g \mapsto -\kappa_g$.



$$\kappa_g = \ddot{\gamma}\cdot(N\times\dot{\gamma})$$

$$\frac{\partial}{\partial s} \mapsto -\frac{\partial}{\partial s} \Rightarrow \kappa_g \mapsto -\kappa_g$$

Hence $\int_S K_G dA = 2\pi(-E+V+F) = 2\pi\chi(S)$. QED.

A corollary

Theorem

Any two diffeomorphic compact surfaces have the same Euler number and (therefore) the same total curvature $\int K_G dA$.

Proof:

- Diffeomorphisms map curves to curves, intersections of curves to intersections of curves, etc.
- Therefore they map triangulations by curvilinear polygons to triangulations by curvilinear polygons, preserving the number of faces, edges, and vertices. QED.

Remark:

- $\mathcal{F}_{II}$ detects curvature of a surface S.
- But $K_G := \det \mathcal{W} = \frac{\det \mathcal{F}_{II}}{\det \mathcal{F}_I}$ depends only on $\mathcal{F}_I$.
- For S compact, then $\int_S K_G dA$ doesn't even depend on local geometry encoded in $\mathcal{F}_I$. It depends only on the *topology* of S.

A further corollary

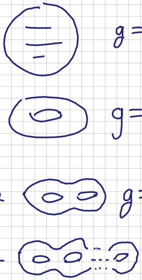
Theorem

Define the genus g of an orientable compact surface S to be the "number of holes". Then

$$\chi(S) = 2 - 2g.$$

- Proof by induction.
- We've proved $g = 0$ and $g = 1$ cases.
- Must prove: If true for g, it's true for $g + 1$.

$\underline{\chi(S) = 2-2g}$



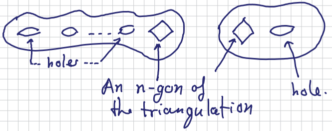
1.	Sphere	$g=0$	$\chi(S^2)=2$
2.	Torus	$g=1$	$\chi(T^2)=0$
3.	Double Torus	$g=2$	$\chi(\Sigma_2)=-2$
4.	Multi-Torus		$\chi(\Sigma_g)=2-2g.$

The induction hypothesis

- Σ_g triangulated n-gons by V vertices, E edges, F faces, with $\chi(\Sigma_g) = F - E + V = 2 - 2g$.
- $\mathbb{T}^2$ triangulated by n-gons V' vertices, E' edges, F' faces, with $\chi(\mathbb{T}^2) = 0$.
- Select one n-gon from each, and glue them together.
- Get $V'' = V + V' - n$ vertices, $E'' = E + E' - n$ edges, $F'' = F + F' - 2$ faces.
- $\chi(\Sigma_{g+1}) = (F + F' - 2) - (E + E' - n) + (V + V' - n) = (F - E + V) + (F' - E' + V') - 2 = \chi(\Sigma_g) - 2$.

1.

2. Triangulate:



3. Glue the n-gons:

Glueing 2 n-gons removes n vertices, n edges, 2 faces

$$\chi(\Sigma_{g+1}) = \chi(\Sigma_g) - 2 + n - n$$
$$= 2 - 2g - 2 = 2 - 2(g+1).$$

Corollary

Corollary

- *No surface diffeomorphic to* $\mathbb{S}^2$ *has* $K_G \leq 0$ *everywhere.*
- *No surface diffeomorphic to a multi-torus has* $K_G \geq 0$ *everywhere.*

Note: We already know that no compact surface in $\mathbb{R}^3$ has $K_G \leq 0$ everywhere.

Proof.

- $\int_{\mathbb{S}^2} K_G \, dA = 2\pi\chi(\mathbb{S}^2) = 4\pi > 0$, so $K_G > 0$ somewhere. This proves part 1.
- $\int_{\Sigma_g} K_G \, dA = 2\pi\chi(\Sigma_g) = 2 - 2g < 0$ for $g > 1$ so $K_G < 0$ somewhere. This proves part 2.
- Remark: Since $K_G > 0$ somewhere on every compact surface embedded in $\mathbb{R}^3$, and $\int_{\mathbb{T}^2} K_G \, dA = 2\pi\chi(\mathbb{T}^2) = 0$, this also proves that $K_G < 0$ somewhere on $\mathbb{T}^2 \subset \mathbb{R}^3$.

□

Lecture 24: Combing hair on compact surfaces

Stationary points of a vector field

Definition

A *stationary point* of a vector field $\mathbf{V}$ is an isolated zero of $\mathbf{V}$. The *index* (or *multiplicity*) of a stationary point p is defined as follows:

- Enclose p within a simple closed unit speed curve γ, traversed counterclockwise.
- Choose any smooth vector field ξ which doesn't vanish on or inside γ.
- Let ψ be the angle from ξ to $\mathbf{V}$ at each point of γ.

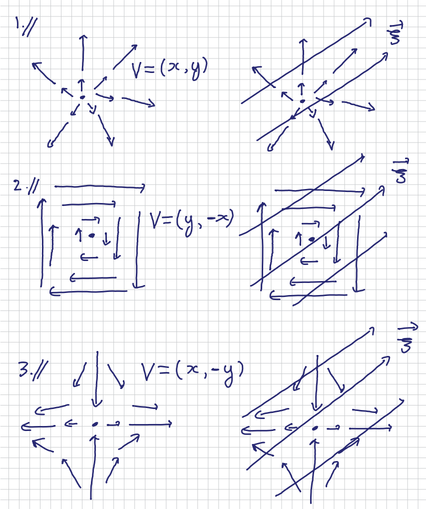
Then the multiplicity $\mu(p)$ of $\mathbf{V}$ at p is

$$\mu(p) = \frac{1}{2\pi}\int_\gamma \frac{d\psi}{ds}\,ds.$$

To understand the definition, note that $\frac{1}{2\pi}\int_\gamma \frac{d\psi}{ds}ds = \psi(b) - \psi(a)$ where $\gamma : [a,b] \to \mathbb{R}^2$ is a closed curve, and $\psi(b) = \psi(a)$ modulo an integer multiple of 2π.

Examples

1. $\mathbf{V} = (x, y)$ has $\mu(p) = 1$.
2. $\mathbf{V} = (y, -x)$ has $\mu(p) = 1$.
3. $\mathbf{V} = (x, -y)$ has $\mu = -1$.



Compute $\mu(p)$ for $\mathbf{V} = (x, -y)$

- Pick ξ to have no zero: $\xi = (1, 0) = \mathbf{e}_1$ will do nicely.
- Compute $\cos\psi = \frac{\mathbf{V}\cdot\xi}{\|\mathbf{V}\|\|\xi\|} = \frac{x}{\sqrt{x^2+y^2}}$.
- Also, $\sin\psi = -\frac{y}{\sqrt{x^2+y^2}}$ (careful of the sign!).
- Encircle p with a simple, closed, unit speed curve traversed counterclockwise, say $\gamma(s) = (\cos s, \sin s)$ (taking coordinates so that p is the origin).
- Then $\cos\psi = \frac{x}{\sqrt{x^2+y^2}} = \frac{\cos s}{\sqrt{\cos^2 s+\sin^2 s}} = \cos s$,
  $\sin\psi = -\frac{y}{\sqrt{x^2+y^2}} = -\frac{\sin s}{\sqrt{\cos^2 s+\sin^2 s}} = -\sin s$ along γ.
- Read off that $\psi(s) = 2\pi - s$.
- Then $\frac{d\psi}{ds} = -1$ and $\mu(p) = \frac{1}{2\pi}\int_0^{2\pi}(-1)ds = -1$.

(N.B. Second-last bullet point also gives $\psi(2\pi) - \psi(0) = -2\pi$ so
$\mu(p) = \frac{1}{2\pi}\int_0^{2\pi}(-1)ds = \frac{1}{2\pi}(\psi(2\pi) - \psi(0)) = -1$ without finding $\frac{d\psi}{ds}$.)

Can you comb the hair on a sphere?

Theorem

If $\mathbf{V}$ *is a smooth vector field on a compact surface* S *with* n *stationary points (i.e.,* n *isolated zeroes)* $p_1, \ldots, p_n$*, then*

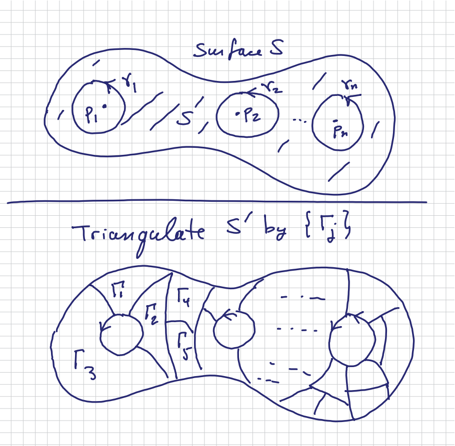
$$\sum_{i=1}^{n} \mu(p_i) = \chi(S).$$

Corollary:

- On a genus g compact surface Σ_g, we have $\chi(\Sigma_g) = 2 - 2g$. From the theorem any smooth vector field $\mathbf{V}$ on Σ_g must have at least one stationary point unless Σ_g is a torus (so $g = 1$).
- To answer the title question, "Not without a bald spot."
- For spheres this generalizes, and is true for all even dimensional spheres $\mathbb{S}^{2n}$ (subsets $x_1^2 + \cdots + x_{n+1}^2 = a^2$, $a > 0$, in $\mathbb{R}^{n+1} \ni (x_1, \ldots, x_{n+1})$). However, you can comb the hair on any odd-dimensional sphere smoothly, without any bald spots (stationary points of the vector field).

The proof

- Surface S, vector field $\mathbf{V}$, stationary points $p_1, \ldots, p_n$.
- Encircle the p_i with (disjoint) unit speed simple closed curves γ_i.
- Let S' be the closure of the region of S outside the γ_i (closure means that the γ_i are included in S').
- Triangulate S' with curvilinear polygons Γ_j. $\mathbf{V}$ has no stationary points in S'.



$$2\pi\chi(S) = \int_S K_G dA = \int_{S'} K_G dA + \sum_{i=1}^{n} \int_{\text{int}(\gamma_i)} K_G dA.$$

The exterior region S'

- Pick ONB $\{\mathbf{e}_1, \mathbf{e}_2\}$ in S', with $\mathbf{e}_1 = \mathbf{V}/\|\mathbf{V}\|$.
- From a previous proof (of the "local" Gauss-Bonnet formula) for a region Σ bounded by curves β_i, we have

  $$\int_\Sigma K_G \, dA = \oint_{\beta_i} \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \, ds,$$

  for $\dot{\mathbf{e}}_2$ the derivative of $\mathbf{e}_2$ along β_i.
- Now let Σ be S' and β_i be $-\gamma_i$. The minus sign is because "counterclockwise about $p_i \in \operatorname{int} \gamma_i$" is clockwise about a point in S'. (Notation: $-\gamma$ is used to indicate reverse orientation, so $\frac{d}{ds} \mapsto -\frac{d}{ds}$, not the negative of the components of γ.)
- Then

  $$\int_{S'} K_G \, dA = -\sum_{i=1}^{n} \oint_{\gamma_i} \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \, ds. \tag{1}$$

The disks $\operatorname{int}\gamma_i$ containing stationary points p_i

- Pick an orthonormal basis $\{\mathbf{E}_1, \mathbf{E}_2\}$.
- As before, get

$$\int_{\operatorname{int}\gamma_i} K_G\, dA = \oint_{\gamma_i} \mathbf{E}_1 \cdot \dot{\mathbf{E}}_2\, ds. \tag{2}$$

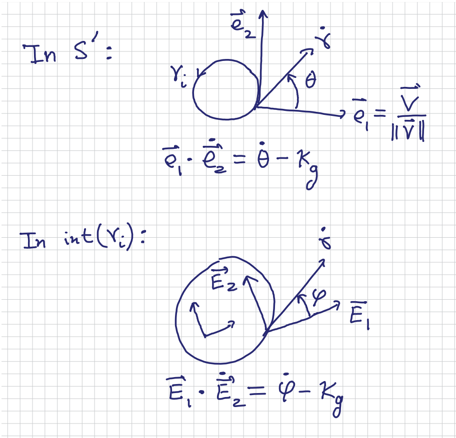
- Combine (1) and (2):

$$\begin{aligned} 2\pi\chi(S) &= \int_S K_G\, dA = \int_{S'} K_G\, dA + \sum_{i=1}^{n} \int_{\operatorname{int}(\gamma_i)} K_G\, dA \\ &= -\sum_{i=1}^{n} \oint_{\gamma_i} \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2\, ds + \sum_{i=1}^{n} \oint_{\gamma_i} \mathbf{E}_1 \cdot \dot{\mathbf{E}}_2\, ds \\ \implies 2\pi\chi(S) &= \sum_{i=1}^{n} \oint_{\gamma_i} \left(\mathbf{E}_1 \cdot \dot{\mathbf{E}}_2 - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \right) ds. \qquad (3) \end{aligned}$$

Put the pieces together

- Use result from "local Gauss-Bonnet" proof. Let κ_g be the geodesic curvature of γ_i, let θ be the angle from $\mathbf{e}_1$ to $\dot{\gamma}$, and let φ be the angle from $\mathbf{E}_1$ to $\dot{\gamma}$.
- Then $\mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 = \dot{\theta} - \kappa_g$.
- Also $\mathbf{E}_1 \cdot \dot{\mathbf{E}}_2 = \dot{\varphi} - \kappa_g$
- $\implies \mathbf{E}_1 \cdot \dot{\mathbf{E}}_2 - \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 = \dot{\varphi} - \dot{\theta}$.
- Equation (3) on the last slide becomes

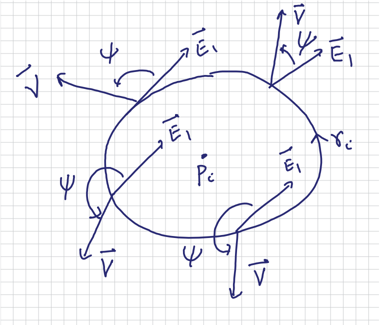
$$2\pi\chi(S) = \sum_{i=1}^{n} \oint_{\gamma_i} \left(\dot{\varphi}(s) - \dot{\theta}(s)\right) ds.$$



Finish the proof

$$2\pi\chi(S) = \sum_{i=1}^{n} \oint_{\gamma_i} \frac{d}{ds}\left(\varphi(s) - \theta(s)\right) ds.$$

- θ is angle from $\mathbf{e}_1$ to $\dot{\gamma}$.
- φ is angle from $\mathbf{E}_1$ to $\dot{\gamma}$.
- Then $\varphi - \theta$ is angle from $\mathbf{E}_1$ to $\mathbf{e}_1$.
- But $\mathbf{e}_1 = \frac{\mathbf{V}}{\|\mathbf{V}\|}$ so $\varphi - \theta = \psi =$ angle from $\mathbf{E_1}$ to $\mathbf{V}$.
- But then $\oint_{\gamma_i} \frac{d}{ds}\left(\varphi(s) - \theta(s)\right) ds = \oint_{\gamma_i} \frac{d\psi}{ds} ds = 2\pi\mu(p_i)$.



We conclude that $2\pi\chi(S) = 2\pi \sum_{i=1}^{n} \mu(p_i)$, so $\chi(S) = \sum_{i=1}^{n} \mu(p_i)$. QED.

Post-mortem

Theorem

If $\mathbf{V}$ *is a smooth vector field on a compact surface* S *with* n *stationary points (i.e.,* n *isolated zeroes)* $p_1, \ldots, p_n$*, then*

$$\sum_{i=1}^{n} \mu(p_i) = \chi(S).$$

- Left-hand side is a statement about vector fields.
- Right-hand side is a topological quantity.
- No local geometry (1FF, 2FF) at all.
- But we needed the tools of local differential geometry to prove this.
- The real power of differential geometry, and its modern incarnation in the form called *geometric analysis*, is its applicability to related but distinct fields.