**Project**

1. To begin, think about the evolution in time of objects like
$$z(t) = x(t) \wedge y(t).$$
We know how to handle this in the case that $x, y$ are both solutions of an equation $\dot{x} = A(t)x$ and that $\dot{z} = A^{[2]}(t)z$. But, if $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$, what about $z(t)$?

2. And then, what would you use it for? What would it mean if $\lim_{t\to\infty} x(t) \wedge y(t) = 0$ for some solutions $x$, $y$? Or for all solutions $x$ and some $y$?

3. This question may be a bit general to start with so there may be some natural relationship that can be assumed between $A(t)$ and $B(t)$.

4. You don't have to stay with homogeneous equations or even linear equations, I suppose.

5. Theorem 1 of the following Muldowney-Samuylova paper considers objects of the form
$$z(t) = x^1(t) \wedge x^2(t) \wedge \cdots \wedge x^k(t)$$
where
$$\dot{x}^i = A(t)x^i + f^i(t), \quad i = 1, \cdots, k.$$
and each of these equations is assumed to have the same linear part. The theorem is an example of what type of information we can get about the equations from the wedges.

6. What would happen if we allow all the equations to have different linear parts as well
$$\dot{x}^i = A^i(t)x^i + f^i(t), \quad i = 1, \cdots, k$$
and what does it say about the equations?

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# DIMENSION PROBLEMS FOR NONHOMOGENEOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. The codimension of stable subspaces of certain function spaces arising from nonhomogeneous linear differential equations is considered. The history of the problem begins with a result on second order linear scalar equations by a French mathematician Milloux in 1934 and continues with Hartman, Coppel, Macki and Muldowney as well as many others in the second half of the twentieth century. The PhD dissertation of Wang (2008) discusses the question in infinite dimensional spaces. This and earlier studies deal exclusively with homogeneous systems. Related questions for nonhomogeneous systems of linear differential equations are considered here. The evolution of $k$-volumes expressed as exterior products of solutions plays a major role in the study.

**Introduction** Consider the linear ordinary differential equation

$$x' = A(t)x + f(t), \quad t \geq 0 \tag{1}$$

where $t \mapsto A(t)$ is a $n \times n$ real matrix valued function, $t \mapsto f(t)$ is a $\mathbb{R}^n$ valued function and $t \in [0, \infty)$. It is assumed that the functions $A$, $f$ are locally Lebesgue integrable on their domains, $A, f \in loc\, L_1[0, \infty)$. A function $t \mapsto x(t) \in \mathbb{R}^n$ is a solution of (1) if it is locally absolutely continuous on $[0, \infty)$ and satisfies (1) almost everywhere.

A solution $x(t) = x(t; \xi, f)$ of (1) is uniquely determined by $x(0) = \xi$ and $f \in loc\, L_1[0, \infty)$. The map

$$(\xi, f) \longmapsto x(\cdot; \xi, f)$$

is linear in $(\xi, f)$. Let

$$\mathcal{F} = \{x \in loc\, AC([0, \infty)) : x' = A(t)x + f(t),\ f \in loc\, L_1[0, \infty)\},$$

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$\mathcal{X}$ be any subspace of $\mathcal{F}$ and

$$\mathcal{X}_0 = \{x \in \mathcal{X} : \lim_{t\to\infty} x(t) = 0\}.$$

We study the codimension of $\mathcal{X}_0$ in $\mathcal{X}$. We begin with a brief review of results on the dimension of zero tending solutions of the homogeneous equation, e.g., when $\mathcal{X}$ is the subspace of $\mathcal{F}$ for which $f = 0$.

In 1934 Milloux [**5**] showed that, if $0 \le a(t)$ is a nondecreasing function for $t \ge 0$, then there exists at least one nontrivial solution $u = u(t)$ of the second order scalar oscillator equation $u'' + a(t)u = 0$ satisfying $\lim_{t\to\infty} u(t) = 0$ if and only if $\lim_{t\to\infty} a(t) = \infty$. The result of Milloux was generalized by Hartman [**3**] and Coppel [**1**] to linear systems of $n$ differential equations

$$y' = A(t)y. \tag{2}$$

Hartman [**3**] proved that if $\lim_{t\to\infty} |y(t)|$ exists and is finite for every solution $y(t)$, where $|\cdot|$ is the Euclidean norm, then there exists at least one nontrivial solution $y = y(t)$ satisfying $\lim_{t\to\infty} y(t) = 0$ if and only if $\int_0^\infty \operatorname{tr}A(s)\,ds = -\infty$. The result was extended to any norm by a different approach in Coppel [**1,** p. 60]. When $a = a(t)$ is continuously differentiable, the theorem of Milloux follows from the theorem of Hartman. It also follows as a simple consequence of Proposition 5 below without the differentiability assumption on the function $a(t)$.

The condition that $\lim_{t\to\infty} |y(t)|$ should exist for all solutions is restrictive in that it depends on the existence of the limit for some norm $|\cdot|$ and can be weakened. Macki and Muldowney [**4**] developed a new approach and replaced the condition by the requirement that for all solutions

$$\begin{array}{ll}(i) & \limsup_{t\to\infty} |y(t)| < \infty, \\ (ii) & \liminf_{t\to\infty} |y(t)| = 0 \Rightarrow \lim_{t\to\infty} y(t) = 0.\end{array} \tag{$L$}$$

This means that solutions are bounded and either tend to 0 or are bounded away from 0. Clearly the condition $(L)$ holds under the Hartman-Coppel condition that $\lim_{t\to\infty} |y(t)|$ exists. But $(L)$ is also satisfied, for example, if (2) is uniformly stable. The condition of Hartman-Coppel does not imply nor is it implied by uniform stability.

Muldowney [**6**] extended these results by showing that, if the condition $(L)$ is satisfied by solutions of the system (2), then the subspace of its solutions that satisfy $\lim_{t\to\infty} y(t) = 0$ has dimension at least $(n - k + 1)$ if and only if all solutions of the associated $k$-th compound equation

$z' = A^{[k]}(t)z$ (see Appendix) satisfy $\lim_{t\to\infty} z(t) = 0$. Equivalently, if the solution space $\mathcal{X}$ of the homogeneous equation satisfies condition $(L)$, then $\operatorname{codim} \mathcal{X}_0 < k$ in $\mathcal{X}$ if and only if $z' = A^{[k]}(t)z$ is asymptotically stable. In the case $k = n$ the compound equation of (2) is $z' = \operatorname{tr} A(t)z$ and this result gives the theorem of Macki and Muldowney.

**Nonhomogeneous linear equations** The following theorem is the main result of this paper and was originally developed in the MSc dissertation of Samuylova [**8**].

**Theorem 1.** *Let $x_i(t)$ be a solution of*

$$x' = A(t)x + f_i(t),$$

*where $f_i \in L_1[0,\infty)$, $i = 1,\dots,k$. Suppose that the homogeneous equation*

$$y' = A(t)y \tag{3}$$

*is uniformly stable and that the $k$-th compound equation*

$$z' = A^{[k]}(t)z$$

*is uniformly stable and asymptotically stable. Then there exist constants $c_1,\dots,c_k$, not all equal zero, such that*

$$\lim_{t\to\infty}(c_1x_1(t) + \cdots + c_kx_k(t)) = 0,$$

*and thus, if $\mathcal{X}$ is any linear subspace of $\{x \in \mathcal{F} : f \in L_1[0,\infty)\}$, then* $\operatorname{codim} \mathcal{X}_0 < k$ *in $\mathcal{X}$.*

**Proposition 2.** *Suppose that the homogeneous equation* (3) *is uniformly stable and $f \in L_1([0,\infty))$. Then the solution space of the nonhomogeneous equation* (1) *satisfies condition* $(L)$.

*Proof.* Let $x(t)$ be a solution of the equation (1), and let $Y(t)$ denote a fundamental matrix of the equation (3). The variation of parameters formula implies that

$$x(t) = Y(t)\left[Y^{-1}(t_0)x(t_0) + \int_{t_0}^{t} Y^{-1}(s)f(s)\,ds\right]$$

for any $t \geq t_0$. From uniform stability of (3) it follows that there exists a positive constant K such that

$$|Y(t)Y^{-1}(s)| \leq K \quad \text{for } t \geq s \geq 0 \tag{4}$$

and consequently

$$|x(t)| \leq K|x(0)| + K \int_{t_0}^{t} |f(s)|\, ds. \tag{5}$$

Therefore $x(t)$ is bounded and so property $(L)(i)$ is satisfied. To verify $(L)(ii)$, let $\varepsilon > 0$. If $\liminf_{t\to\infty} |x(t)| = 0$, there exists $t_0$ such that $|x(t_0)| < \varepsilon/(2K)$ and also, since $f \in L_1([0,\infty))$, such that $\int_{t_0}^{t} |f(s)|\, ds < \varepsilon/(2K)$ if $t_0 \leq t$. Therefore (5) implies $|x(t)| < \varepsilon$ if $t > t_0$. It follows that $\lim_{t\to\infty} |x(t)| = 0$ and so $x(t)$ satisfies $(L)(ii)$. □

**Proposition 3.** *Suppose that the homogeneous equation* (3) *is uniformly stable and asymptotically stable and that $f \in L_1([0,\infty))$. Then all solutions $x = x(t)$ of the nonhomogeneous equation* (1) *satisfy* $\lim_{t\to\infty} x(t) = 0$.

*Proof.* The variation of parameters formula for solutions of (1) gives

$$x(t) = Y(t)\left[Y^{-1}(0)x(0) + \int_{0}^{T} Y^{-1}(s)f(s)\, ds + \int_T^t Y^{-1}(s)f(s)\, ds\right]. \tag{6}$$

Because (3) is asymptotically stable, $\lim_{t\to\infty} |Y(t)| = 0$ and therefore

$$\lim_{t\to\infty} Y(t)\left[Y^{-1}(0)x(0) + \int_{0}^{T} Y^{-1}(s)f(s)\, ds\right] = 0 \tag{7}$$

for each $T > 0$. Uniform stability of (3), (4) and $f \in L_1([0,\infty))$ imply that, if $\varepsilon > 0$, $T$ can be chosen sufficiently large that

$$\left|Y(t)\int_0^T Y^{-1}(s)f(s)\, ds\right| \leq K \int_{T}^{t} |f(s)|\, ds < \varepsilon \tag{8}$$

if $t \geq T$. Thus, by (6), (7) and (8), $\lim_{t\to\infty} x(t) = 0$. □

**Remark 4.** If the homogeneous equation (3) is asymptotically stable and, for some $f(t)$, there is a solution $x_0(t)$ of the nonhomogeneous equation (1) such that $\lim_{t\to\infty} x_0(t) = 0$, then all solutions of the equation (1) satisfy $\lim_{t\to\infty} x(t) = 0$. This follows from the fact that for any solution $x(t)$ of the nonhomogeneous equation (1) $x(t) - x_0(t)$ is a solution of the corresponding homogeneous equation (3). When the equation (3) is asymptotically stable the Remark implies that, for a fixed function $f(t)$, either all solutions of (1) tend to 0 as $t \to \infty$ or none does.

The proof of the main theorem relies on the following proposition regarding linear spaces of $R^n$-valued functions on $[0,\infty)$, [**7**]. This result was extended by Wang to linear spaces of functions $x : [0,\infty) \to V$, where $V$ is any vector space [**9**]. We give the proof of the proposition, as it is basic for Theorem 1.

**Proposition 5** (Muldowney, 1990)**.** *Let $\mathcal{X}$ be a linear space of functions $x : [0,\infty) \to \mathbb{R}^n$ that satisfies $(L)$. Then*

$$\operatorname{codim} \mathcal{X}_0 < k \Longleftrightarrow {\mathcal{X}^{(k)}}_0 = \mathcal{X}^{(k)}.$$

Here $\mathcal{X}^{(k)}$ denotes the *k-th exterior power* of $\mathcal{X}$, $1 \leqslant k \leqslant n$, which is defined by

$$\mathcal{X}^{(k)} = sp\,\{x^1 \wedge \cdots \wedge x^k : x^i \in \mathcal{X}\}.$$

and $\mathcal{X}_0$ and ${\mathcal{X}^{(k)}}_0$ denote subspaces of $\mathcal{X}$ and $\mathcal{X}^{(k)}$, respectively, defined by

$$\mathcal{X}_0 = \big\{x \in \mathcal{X} : \lim_{t\to\infty} x(t) = 0\big\},$$

$${\mathcal{X}^{(k)}}_0 = \big\{w \in \mathcal{X}^{(k)} : \lim_{t\to\infty} w(t) = 0\big\}.$$

*Proof.* Suppose $\operatorname{codim} \mathcal{X}_0 < k$. Let $x^1, x^2, \ldots, x^k$ be any elements of $\mathcal{X}$. Then there is a nontrivial $x \in sp\,\{x^1, x^2, \ldots, x^k\}$ such that $\lim_{t\to\infty} x(t) = 0$. Without loss of generality

$$x(t) = c_1 x_1(t) + \cdots + c_k x_k(t),$$

where $c_1, \ldots, c_k \in \mathbb{R}$, $c_1 \neq 0$. Then $y = x^1 \wedge x^2 \wedge \cdots \wedge x^k = \frac{1}{c_1} x \wedge x^2 \wedge \cdots \wedge x^k$ and inequality (12) from the Appendix imply $|y(t)| \leq \frac{c}{|c_1|}|x(t)||x^2(t)| \cdots |x^k(t)|$. From here it follows that $\lim_{t\to\infty} y(t) = 0$, by $(L)(i)$. Thus ${\mathcal{X}^{(k)}}_0 = \mathcal{X}^{(k)}$.

Conversely, suppose ${\mathcal{X}^{(k)}}_0 = \mathcal{X}^{(k)}$. Let $x^1, x^2, \dots, x^k$ be any elements of $\mathcal{X}$. Then $y = x^1 \wedge x^2 \wedge \cdots \wedge x^k$ satisfies $\lim_{t\to\infty} y(t) = 0$. Now $(L)(i)$ implies that there exists a sequence $t_i \to \infty$ such that $\lim_{i\to\infty} X(t_i) = C$ exists, where $X$ is the $n \times k$ matrix whose columns are $x^1, x^2, \dots, x^k$. We have $C^{(k)} = \lim_{i\to\infty} \mathcal{X}^{(k)}(t_i) = 0$ so that $\mathrm{rk}C < k$ and there exists a nonzero vector $\xi \in \mathbb{R}$ such that $C\xi = 0$. Therefore $x = X\xi$ satisfies $\lim_{i\to\infty} x(t_i) = \lim_{i\to\infty} X(t_i)\xi = C\xi = 0$, which implies $\lim_{t\to\infty} x(t) = 0$, by $(L)(ii)$. Thus $\operatorname{codim} \mathcal{X}_0 < k$. $\square$

Proposition 5 implies that, for any subspace $\mathcal{X}$ of $\mathcal{F}$ that satisfies $(L)$, the set $\mathcal{X}_0$ of solutions $x(t)$ in $\mathcal{X}$ that tend to zero at infinity has codimension at most $k$ in $\mathcal{X}$ if and only if

$$\lim_{t\to\infty} x_1(t) \wedge \cdots \wedge x_k(t) = 0$$

for $x_i(t) \in \mathcal{X}$, $i = 1, \dots, k$. If $x_i' = A(t)x_i + f_i(t)$, then, by the property (13) of $A^{[k]}(t)$ (see Appendix),

$$\begin{aligned}
(x_1 \wedge \cdots \wedge x_k)' &= \sum_1^k x_1 \wedge \cdots \wedge x_i' \wedge \cdots \wedge x_k \\
&= \sum_1^k x_1 \wedge \cdots \wedge (A(t)x_i + f_i(t)) \wedge \cdots \wedge x_k \\
&= \sum_1^k x_1 \wedge \cdots \wedge A(t)x_i \wedge \cdots \wedge x_k \\
&\qquad + \sum_1^k x_1 \wedge \cdots \wedge f_i(t) \wedge \cdots \wedge x_k \\
&= A^{[k]}(t)\,(x_1 \wedge \cdots \wedge x_k) + \sum_1^k x_1 \wedge \cdots \wedge f_i(t) \wedge \cdots \wedge x_k,
\end{aligned}$$

and $w = x_1 \wedge \cdots \wedge x_k$ satisfies

$$w' = A^{[k]}(t)w + \sum_1^k x_1 \wedge \cdots \wedge f_i(t) \wedge \cdots \wedge x_k. \tag{9}$$

*Proof of Theorem 1.* Let $\mathcal{X} = \{x \in \mathcal{F} : f \in L_1[0,\infty)\}$, and let $x_i(t) \in \mathcal{X}$, $i = 1, \dots, k$. By Proposition 2, $\mathcal{X}$ satisfies $(L)$. To apply Proposition 5, we should demonstrate that solutions $w = w(t)$ of (9) tend to zero at infinity. Since $\mathcal{X}$ satisfies $(L)(i)$, there exists a positive constant $K$ such that

$$|x_1(s) \wedge \cdots \wedge f_i(s) \wedge \cdots \wedge x_k(s)| \leq c|x_1(s)| \cdots |f_i(s)| \cdots |x_k(s)| \\ \leq K|f_i(s)|, \quad i = 1, \dots, k,$$

where $c$ is a constant depending on the norm (see (12), Appendix). Therefore, if $f_i \in L_1([0,\infty))$, $i = 1, \dots, k$, then $\sum_1^k x_1 \wedge \cdots \wedge f_i \wedge \cdots \wedge x_k \in L_1([0,\infty))$. Thus, since (3) is uniformly stable and asymptotically stable, Proposition 3 implies $\lim_{t\to\infty} w(t) = 0$. Thus, by applying Proposition 5, we conclude that codim $\mathcal{X}_0 < k$ in $\mathcal{X}$, or, equivalently, there exist constants $c_1, \dots, c_k$, not all equal zero, such that $\lim_{t\to\infty}(c_1 x_1(t) + \cdots + c_k x_k(t)) = 0$. □

Consider the scalar second order differential equation

$$v'' + a(t)v = e(t). \tag{10}$$

If $a$ is locally Lebesgue integrable, then the locally absolutely continuous solution $v = v(t; v_0, v_1, e)$ is uniquely determined by $v(0) = v_0$, $v'(0) = v_1$ and $e$ provided $e$ is also locally integrable. The solution depends linearly on $(v_0, v_1, e)$.

**Corollary 6.** *Suppose that* $0 < a$ *is nondecreasing on* $[0,\infty)$. *Then for any solutions* $v_i$ *of*

$$v'' + a(t)v = e_i(t),$$

*where* $e_i/\sqrt{a} \in L_1[0,\infty)$, $i = 1, 2$, *there exist constants* $c_1$ *and* $c_2$, $|c_1| + |c_2| \neq 0$, *such that*

$$\lim_{t\to\infty}(c_1 v_1(t) + c_2 v_2(t)) = 0$$

*if and only if* $\lim_{t\to\infty} a(t) = \infty$.

The corollary follows from Theorem 1 by reducing the equation (10) to an equivalent system (1), where

$$x = \begin{pmatrix} v \\ v'/a^{\frac{1}{2}} \end{pmatrix}, \quad A = \begin{bmatrix} 0 & a^{\frac{1}{2}} \\ -a^{\frac{1}{2}} & -a'/2a \end{bmatrix}, \quad f = \begin{pmatrix} 0 \\ e/a^{\frac{1}{2}} \end{pmatrix}.$$

In the case of a monotone nondifferentiable function $a(t)$ an approximation of $a(t)$ by $C^1$ functions can be applied.

Corollary 6 includes the theorem of Milloux mentioned in the Introduction. Though it cannot be reduced to the result of Milloux, for example if $v_1(t) = v(t;0,0,e_1)$ and $v_2(t) = v(t;0,0,e_2)$ are both taken with homogeneous initial conditions. It was proved later by Armellini, Hartman, Sansone and Tonelli that if $\lim_{t\to\infty} a(t) = \infty$ and $\log a(t)$ has "regular growth," then every solution $u = u(t)$ of the equation

$$u'' + a(t)u = 0 \tag{11}$$

must satisfy $\lim_{t\to\infty} u(t) = 0$."Regular growth" means roughly that the growth does not occur only in $t$-sets that are meager as $t \to \infty$. (see, for example, [**2**])

Milloux showed that not all solutions of (11), that are solutions $v = v(t;v_0,v_1,0)$ of (10), necessarily tend to zero [**5**]. In the following example we consider the complementary linear subspace of solutions $v(t;0,0,e)$ with homogeneous initial conditions $v(0) = v_0$, $v'(0) = 0$. It follows from Corollary 6 that, if $e_1$, $e_2$ are any two functions with $e_1/\sqrt{a} \in L_1([0,\infty))$, $e_2/\sqrt{a} \in L_1([0,\infty))$, there exist constants $c_1$, $c_2$ such that $\lim_{t\to\infty} v(t;0,0,c_1e_1 + c_2e_2) = 0$ or, equivalently,

$$v'' + a(t)v = c_1e_1(t) + c_2e_2(t), \quad v(0) = v'(0) = 0$$

implies $\lim_{t\to\infty} v(t) = 0$. We show by an example that the corollary is nontrivial on this subspace also by constructing a function $0 < a(t)$ which is nondecreasing on $[0,\infty)$ and a function $e(t)$ such that $e/\sqrt{a} \in L_1([0,\infty))$ and there exists a solution $v = v(t)$ of (10) for which $\lim_{t\to\infty} v(t)$ does not exist.

**Example.** First, we construct a function $a(t)$ such that there exist solutions $u_1(t)$ and $u_2(t)$ of the homogeneous equation (11) for which $\lim_{t\to\infty} u_1(t)$ does not exist and $\lim_{t\to\infty} u_2(t) = 0$.

Let $a(t)$ be the step-function defined by $a(0) = 1$ and $a(t) = j^2$ for $t \in (t_{j-1}, t_j]$, $j = 1, 2, \dots$, where $t_0 = 0$, $t_j = \frac{\pi}{2} + 2\pi(j-1)$, $j \ge 1$ (see Figure 1). If $u_1(0) = 0$, $u_1'(0) = 1$, then $u_1(t) = \sin(t)$ for $t \in [0, \frac{\pi}{2}]$ and $u_1(t) = d_1 \sin(2t) + d_2 \cos(2t)$ for $t \in (\frac{\pi}{2}, \frac{\pi}{2} + 2\pi]$, where $d_1, d_2$ are two constants. As $u_1(\frac{\pi}{2}) = 1$ and $u_1'(\frac{\pi}{2}) = 0$, we get $u_1(t) = -\cos(2t)$ for $t \in (\frac{\pi}{2}, \frac{\pi}{2} + 2\pi]$. Since the length of all the intervals is $2\pi$, by the same argument, we have

$$u_1(t) = \begin{cases} (-1)^k \sin(jt), & (t_{j-1}, t_j],\ j = 2k+1, \\ (-1)^k \cos(jt), & (t_{j-1}, t_j],\ j = 2k. \end{cases}$$

Hence $u_1(t_j) = 1$, $j = 1, 2, \dots$, and $\lim_{t\to\infty} u_1(t)$ does not exist. Similarly, by choosing $u_2(0) = 1$ and $u_2'(0) = 0$ we get

$$u_2(t) = \begin{cases} \frac{(-1)^k}{j}\sin(jt), & (t_{j-1}, t_j],\ j = 2k, \\ \frac{(-1)^k}{j}\cos(jt), & (t_{j-1}, t_j],\ j = 2k+1. \end{cases}$$

Therefore, $\lim_{t\to\infty} u_2(t) = 0$.

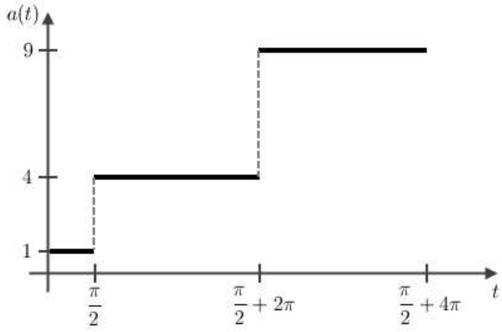


FIGURE 1: The graph of function $a(t)$.

The solution $v(t)$ of the nonhomogeneous equation (10) can be written in the form

$$v(t) = u_1(t)\int_0^t u_2(s)e(s)\,ds - u_2(t)\int_0^t u_1(s)e(s)\,ds.$$

Let $F_1(t) = u_1(t)\int_0^t u_2(s)e(s)\,ds$ and $F_2(t) = u_2(t)\int_0^t u_1(s)e(s)\,ds$. Define $e(t) = 1$ for $t \in [t_0, t_1]$ and $e(t) = (-1)^{k+1}/j$ for $t \in (t_{j-1} + \pi k/j, t_{j-1} + \pi(k+1)/j]$, $k = 0, \dots, 2j-1$, $j \ge 2$ (see Figure 2). Then $e/\sqrt{a} \in L_1([0,\infty))$. Hence

$$\begin{aligned} \big|F_1(t_n)\big| &= \left|u_1(t_n)\int_0^{t_n} u_2(s)e(s)\,ds\right| = \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \frac{|u_2(s)|}{j}\,ds \\ &= \sum_{j=1}^n \frac{2}{j}\int_0^{\pi/j} \sin(js)\,ds = 4\sum_{j=1}^n \frac{1}{j^2} = \frac{2}{3}\pi^2. \end{aligned}$$

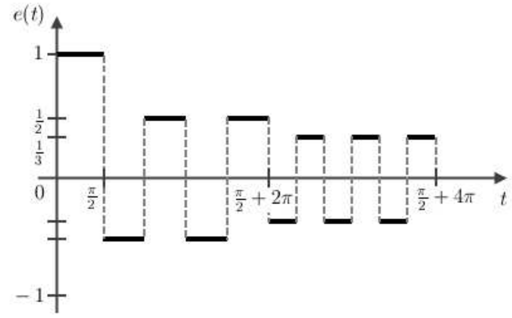


FIGURE 2: The graph of function $e(t)$.

Since $u_2(t_n) = 0,\ n = 1, 2, \ldots,$ we have

$$F_2(t_n) = u_2(t_n) \int_0^{t_n} u_1(s)e(s)\, ds = 0.$$

Then $\lim_{t\to\infty} v(t_n) = 2\pi^2/3$. Similarly, $F_1(t_n + \pi/(2(n+1))) = 0$,

$$\lim_{n\to\infty} \left| F_2\left(t_n + \frac{\pi}{2(n+1)}\right)\right| \\ = \lim_{n\to\infty} \left( \int_0^{\pi/2} \sin(s)\, ds + \frac{1}{n+1} \int_0^{\pi/(n+1)} \sin(ns)\, ds \right) = 1,$$

$n = 1, 2, \ldots.$ So, $\lim_{n\to\infty} v(t_n + \frac{\pi}{2(n+1)}) = 1$. Therefore, $\lim_{t\to\infty} v(t)$ does not exist.

**Appendix** This appendix lists some properties of compound matrices used throughout the paper. A survey on compound matrices with further references may be found in [**7**].

**Definition 7.** If $u^1, \ldots, u^k \in \mathbb{R}^n$, $1 \le k \le n$, let $u_i^j$ be the $i$ component of $u^j$ with respect to the standard basis in $\mathbb{R}^n$. For $1 \le i \le \binom{n}{k}$, let $(i)$ denote the $i$-th element in the lexicographic ordering of the $k$-tuples

$(i) = (i_1, \dots, i_k)$ such that $1 \le i_1 < \cdots < i_k \le n$. Then the *exterior product*

$$\alpha = u^1 \wedge \cdots \wedge u^k$$

may be represented as the element of $\mathbb{R}^N$, $N = \binom{n}{k}$, whose $i$ component is

$$\alpha_i = u^{1\dots k}_{i_1\dots i_k} = \det [u^j_{i_r}], \quad 1 \le r, j \le k.$$

Traditionally a vector $u$ is thought of as a directed line segment. Then the vector $\alpha$ can be geometrically interpreted as the oriented parallelepiped spanned by $u^1, \dots, u^k$. If $u^1, \dots, u^k \in \mathbb{R}^n$ and $|\cdot|$ is any norm on $\mathbb{R}^N$, then there is a positive constant $c$ depending on the norm such that

$$|u^1 \wedge \cdots \wedge u^k| \le c|u^1| \cdots |u^k|. \tag{12}$$

Let $A = [a^j_i]$ be a $m \times n$ matrix, and let $a^{j_1\dots j_k}_{i_1\dots i_k} = \det [a^{j_s}_{i_r}]$, $1 \le r, s \le k$, where $1 \le i_1 < \cdots < i_k \le m$, $1 \le j_1 < \cdots < j_k \le n$.

**Definition 8.** The $k$-th *multiplicative compound* $A^{(k)}$ of $A$ is the $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries, written in lexicographic order, are $a^{j_1\dots j_k}_{i_1\dots i_k}$.

The main properties of the multiplicative compounds are

- If $u^1, \dots, u^k \in \mathbb{R}^n$, then

$$A^{(k)}(u^1 \wedge \cdots \wedge u^k) = Au^1 \wedge \cdots \wedge Au^k.$$

- The previous property implies the multiplicative identity

$$(AB)^{(k)} = A^{(k)} B^{(k)}$$

  for any matrices $A$ and $B$ of dimension consistent with the matrix multiplication. It is known as the *Binet-Cauchy Identity.* This is the motivation for the term *multiplicative compound.*

**Definition 9.** If $m = n$, the $k$-th *additive compound* $A^{[k]}$ of $A$ is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined by

$$A^{[k]} = \frac{d}{dt}\,(I + tA)^{(k)}\Big|_{t=0} = \lim_{t\to 0} \frac{1}{t}\left[(I + tA)^{(k)} - I^{(k)}\right].$$

The main properties of the additive compounds are

- If $u^1, \dots, u^k \in \mathbb{R}^n$, then

$$A^{[k]}(u^1 \wedge \cdots \wedge u^k) = \sum_{r=1}^{k} u^1 \wedge \cdots \wedge Au^r \wedge \cdots \wedge u^k. \tag{13}$$

- If $A$ and $B$ are $n \times n$ matrices, then the previous property implies the additive identity

$$(A + B)^{[k]} = A^{[k]} + B^{[k]}.$$

  Hence the term *additive compound.*

  For a general $n \times n$ matrix $A$

$$A^{(1)} = A^{[1]} = A, \ A^{(n)} = \det A \text{ and } A^{[n]} = \mathrm{tr} A.$$

In the case of a $2 \times 2$ matrix $A^{(2)} = a^{12}_{12} = \det A$ and $A^{[2]} = a^1_1 + a^2_2 = \mathrm{tr} A$.

The connection between compound matrices and differential equations is as follows. Consider a linear differential equation

$$y' = A(t)y \tag{14}$$

where $t \mapsto A(t)$ is a $n \times n$ matrix valued function. Suppose that $y^1(t), \dots, y^k(t)$ are solutions of (14). Then $z(t) = y^1(t) \wedge \cdots \wedge y^k(t)$ satisfies

$$\begin{aligned} z'(t) &= \sum_{i=1}^{k} y^1(t) \wedge \cdots \wedge (y^i(t))' \wedge \cdots \wedge y^k(t) \\ &= \sum_{i=1}^{k} y^1(t) \wedge \cdots \wedge A(t)y^i(t) \wedge \cdots \wedge y^k(t) \\ &= A^{[k]}(t)z(t). \end{aligned}$$

The equation

$$z' = A^{[k]}(t)z \tag{15}$$

is called the *k-th compound equation* of (14). If $Y(t)$ is a fundamental matrix of (14), then its $k$-th multiplicative compound $Y^{(k)}(t)$ is a fundamental matrix of (15). Indeed, any solution of (14) can be written as $Y(t)c$ where $c \in \mathbb{R}^n$. Let $c^1, \dots, c^k \in \mathbb{R}^n$. Then $Y(t)c^1, \dots, Y(t)c^k$ are solutions of (14) and $z(t) = Y(t)c^1 \wedge \cdots \wedge Y(t)c^k = Y^{(k)}(t)c^1 \wedge \cdots \wedge c^k$ is a solution of (15). Therefore $Y^{(k)}(t)$ is a fundamental matrix of (15).

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